Exercises 7

1. Consider the following 4x4 matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & -5 & -5 & -5 \\ -5 & 3 & -5 & -5 \\ -5 & -5 & 3 & -5 \\ -5 & -5 & -5 & 3 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$

   a) Find $A_{2,3}$ the entry in the second column of the third row of A;

   b) Find the third row of B;

   c) Find the second column of C;

   d) Show that $AB \neq BA$;

   e) Calculate $\frac{1}{4}B^2$;

   f) Calculate ABC;

   g) Calculate C^T;

   h) Calculate $A^T + A$;

   i) Calculate $|A|$, $|B|$, $|C|$;

   j) Calculate B^{-1};

   k) Show that $C^{-1}C = \mathbb{I}$;

   l) Show that $(5C)^{-1} = \frac{1}{5}C^{-1}$ (note, you may need to use `np.allclose` to overcome numerical inaccuracies).

2. Solve the following systems of linear equations:

   a)

$$\begin{aligned} 2x_1 + 3x_2 &= 4 \\ 3x_1 + 5x_2 &= 12 \end{aligned}$$

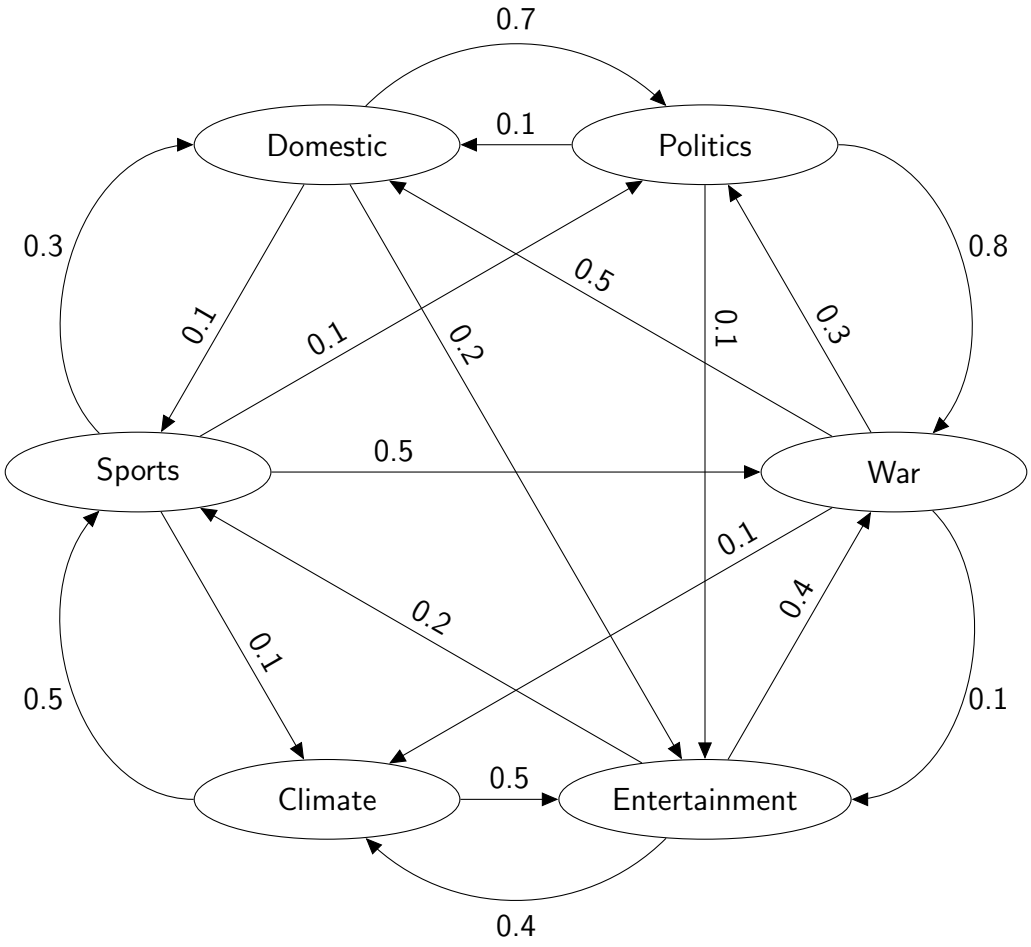
   b)

$$\begin{aligned} 2x + 3y + z - w &= 4 \\ 6x + 5y + 10z + 3w &= 12 \\ x + y + z + w &= 8 \\ 9y - 10x + 5w &= 10 \end{aligned}$$

c)

$$\begin{aligned}
\frac{1}{2}a + \frac{1}{4}b &= 20 \\
\frac{1}{2}a + \frac{1}{5}b + \frac{3}{2}c &= 31 \\
2a + \frac{1}{2}b + \frac{9}{4}c &= 88 \\
3a - b + \frac{1}{8}c &= 71 \\
5b - 8c &= 36
\end{aligned}$$

3. A 24 hour news channel airs different hour-long segments at various times throughout the day. These segments are chosen randomly Dometic news, Politics, War news, Entertainment, Climate updates, and Sports. The next hour's segment is chosen randomly, but based on the current hour's segment. This forms a Markov chain, shown below:



For example, if the channel is currently showing Domestic news, then there is probability 0.7 that the next hour's news will be about Politics.

a) Represent this Markov chain as a probability matrix with Numpy.

b) I turn on the channel and the current segment is Entertainment. What is the probability that I will see a War segment when I return in two hours?

c) In the long run, what proportion of the time is devoted to each type of news segment?

d) How much more War news is aired in comparison to Sports news?

e) Domestic, Politics, and War news cost £10 per hour to air, Climate news costs £20 per hour, Entertainment costs £50 per hour, and Sports costs £60 per hour to air. How much does the channel cost to run on average for 24 hours? (Hint: use `np.dot` for the dot product of two vectors).

4. Consider an $M/M/1$ queue with arrival rate $\Lambda = 25$ and service rate $\mu = 28$. Assume that the maximum length of the queue is 100 people.

a) Create a Numpy matrix representing the transition rate matrix for this queue. (Hint: A state is an integer representing the number of customers present in the queue, so there are 101 states, and so we would expect a (101×101) transition rate matrix, which is too big to write out by hand. Write a function that takes in two states and returns the appropriate transition rate, and use this in a list comprehension to create the transition rate matrix.)

b) If there are no customers in the system now, what is the probability of there being 7 or more customers in the system after 0.8 time units?

c) Find the steady state probabilities of being in each state of the queue.

d) Plot the states against the steady state probability of being in each state of the queue.

e) What is the expected number of customers in the queue? (Hint: use `np.dot` for the dot product of two vectors).