Do not turn this page over until instructed to do so by the Invigilation Supervisor

Academic Year: Mock Exam 1

Examination Period: Mock Exam 1

Module Code: MA2601

Examination Paper Title: Operational Research

Duration: 3 hours

Please read the following information carefully:

Structure of Examination Paper:

- There are **8** pages including this page.
- There are **TWO** sections.
- There are **10** questions in total.
- There are no appendices.
- The maximum mark for the examination paper is 150 and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Instructions for completing the examination:

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **ALL** questions from **Section A**, and **ALL** questions from **Section B**.
- Each question should be answered on a separate page.

You will be provided with / or allowed:

- **ONE** answer book.
- Squared graph paper.
- The **use of calculators is permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.

Section A

(Answer ALL Questions)

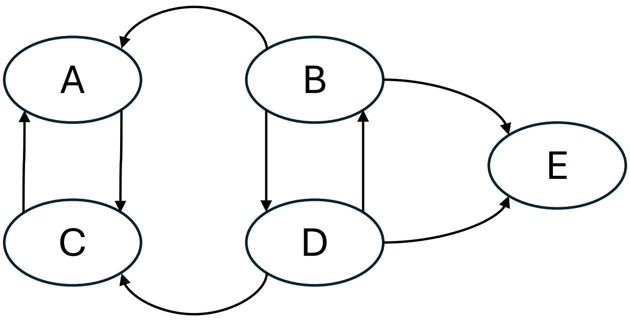
1. Passengers arrive at an airport check-in counter randomly, following a Poisson process with a mean rate of 15 per hour.

   (a) Find the probability of having at least 3 arrivals in an interval of 4 minutes. **[5]**

   (b) What is the mean time between consecutive arrivals? **[2]**

   (c) What is the probability that there is at least a 40-second gap between two consecutive arrivals? **[3]**

2. (a) Consider the continuous-time Markov chain on five states visualised below, where transitions are drawn only if there is a non-zero transition rate between two states.



   i. Write down all its irreducible classes, stating whether they are closed or not closed. **[3]**

   ii. Label each state as either Recurrent, Transient, or Absorbing. **[5]**

   (b) Consider the discrete-time Markov chain on three states defined by the transition probability matrix

$$P = \begin{pmatrix} 0.0 & 0.5 & x \\ 0.1 & 0.9 & 0.0 \\ 0.4 & 0.0 & 0.6 \end{pmatrix}$$

   i. Give the value of x that ensures that P is a valid transition probability matrix for the Markov chain. **[2]**

   ii. Find the steady-state probability vector for this Markov chain. **[7]**

3. At an automated car wash, cars arrive randomly, following a Poisson process with rate $\lambda = 12$ per hour. There are three machines that can wash cars in parallel, and each machine takes exactly 10 minutes to wash a car. If all three machines are busy, then cars queue up to use the machines, and there is room for 6 cars to wait at any one time, if there are 6 cars queueing then arriving cars are turned away and look elsewhere for a car wash. Waiting cars are called to the car washing machines in the order in which they arrived.

   (a) Using Kendall's notation, describe the system as a queue. **[5]**

   The car wash decides to expand the waiting area, and there is now so much queueing capacity that it can be modelled as infinite.

   (b) What is the traffic intensity of the system? **[1]**

   (c) Is this a stable system? **[1]**

   (d) On average there are 1.02 customers in the system, what is the average amount of time spent in the system? **[2]**

4. Use the Simplex method to solve the following linear programming problem:

   Minimise $3X_1 - X_2 - 5X_3$
   subject to

$$\begin{aligned} -X_1 + 5X_2 + X_3 &\le 2 \\ 5X_1 + X_2 &\le 7 \\ -2X_2 + X_3 &\le 4 \\ X_1, X_2, X_3 &\ge 0 \end{aligned}$$

[8]

5. The following tableau is obtained by carrying out the Simplex algorithm:

	X_1	X_2	s_1	s_2	s_3
$10/3$	0	8	1	$-2/3$	0
$10/3$	1	-2	0	$1/3$	0
$8/3$	0	3	0	$-1/3$	1
$10/3$	0	0	0	$1/3$	0

   (a) Read off a solution given by this tableau. **[1]**

   (b) Pivot one more time to find another optimal solution. **[4]**

   (c) Write down the set of *all* optimal solutions as a parametrisation of a straight line segment. **[2]**

   (d) If the value of X_2 is fixed as $X_2 = 1/12$, what value must X_1 take for the solution to remain optimal? **[3]**

6. A large community club has 2,200 members, consisting of 1,150 children, 450 adults, and 600 old age pensioners. It is organising an annual theatre trip for its members. There are four plays:

- "Romeo and Juliet", with 730 tickets available,
- "As You Like It", with 60 tickets available,
- "King Lear", with 410 tickets available, and
- "The Merchant of Venice", with 1,000 tickets available.

The price of each play is different, and there are different prices for children, adults, and old age pensioners, as shown in the table below:

	Children	Adults	Pensioners
Romeo and Juliet	£2	£5	£3
As You Like It	£11	£15	£10
King Lear	£7	£12	£4
The Merchant of Venice	£8	£9	£6

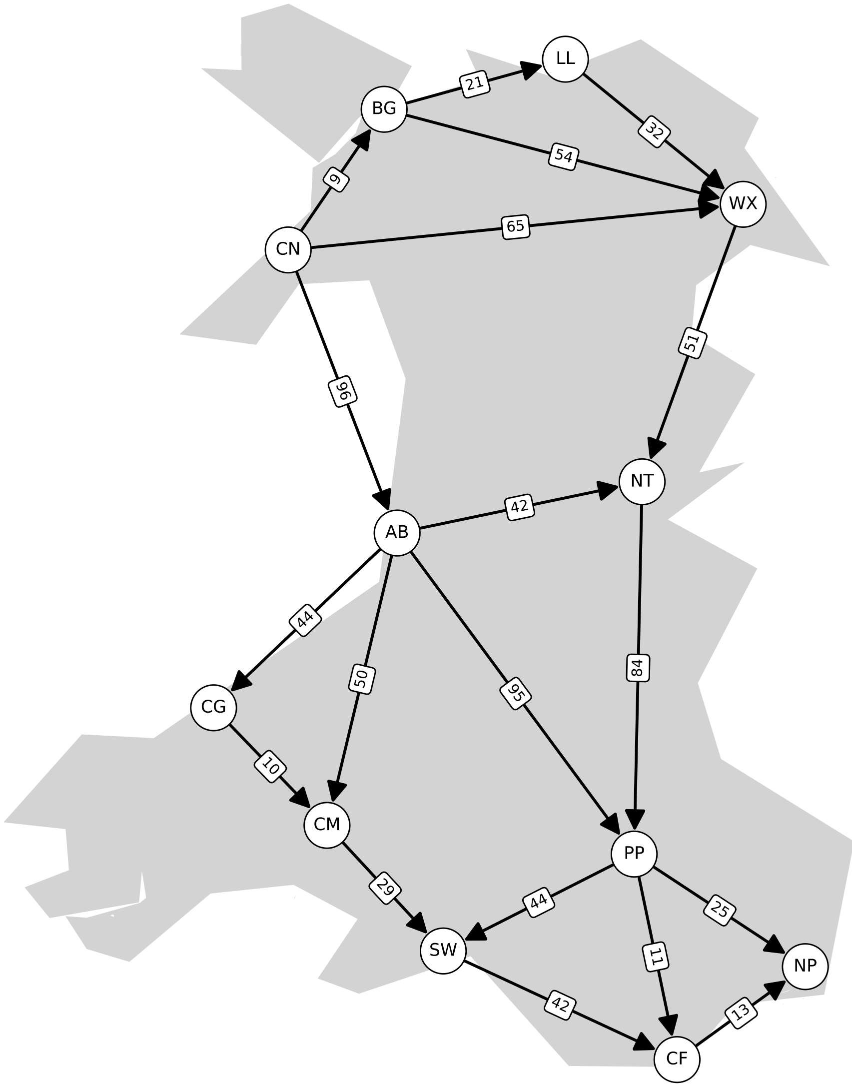
Each member must be bought a ticket to one of these plays.

(a) Find a feasible basic allocation of tickets using the minimum cost method. **[4]**

(b) Use the Stepping-Stone algorithm to find an allocation that minimises the total cost of the tickets. **[6]**

(c) The theatre showing 'As You Like It' want to sell the group more adult tickets, and decide to put on a promotion. What range of values should they sell adult tickets for in order to change the group's optimal allocation? **[3]**

7. A student is stranded in Caernarfon (CN), and is trying to return home to Newport (NP) via public transport, however there is no direct bus route between the two towns. They have identified the following 19 intermediate bus routes, requiring changes in either Bangor (BG), Llandudno (LL), Wrexham (WX), Aberystwyth (AB), Newtown (NT), Cardigan (CG), Carmarthen (CM), Pontypridd (PP), Swansea (SW) or Cardiff (CF). The routes and their distances are given on the map below:



Use the value iteration algorithm to identify the shortest bus journey from Caernarfon (CN) to Newport (NP). **[8]**

Section B
(Answer ALL questions)

8. (a) Consider an $M/M/2/4$/FIFO queue, with arrival rate λ and service rate μ. This can be modelled as a continuous-time Markov chain on five states representing the number of customers present in the system.

   i. Draw a visualisation of this Markov chain. **[3]**

   ii. Give the transition rate matrix, Q. **[2]**

   iii. Find the steady-state probabilities $P_0, P_1, \ldots, P_4$ in terms of λ and μ. **[8]**

   iv. Now using $\lambda = 5$ and $\mu = 5$, find the average number of customers in the system. **[3]**

   (b) Consider an $M/M/1$ queue with arrival rate λ, service rate μ, and traffic intensity $\rho = \lambda/\mu$. The steady-state probabilities are given by:

$$P_k = \rho^k P_0$$
$$P_0 = 1 - \rho$$

   i. What is the probability that an arriving customer has to wait? **[1]**

   ii. Using the above, show that the expected number of customers in the system is given by

$$L = \frac{\rho}{1 - \rho}$$

   . **[4]**

   iii. Using Little's law, show that the expected time spent in the system is given by

$$W = \frac{1}{\mu - \lambda}$$

   . **[4]**

9. Señora Martinez has inherited a number of vineyards in the Sherry Triangle of Spain. She now has 500 acres growing palomino grapes, and 100 acres growing moscatel grapes. Each acre of vineyard produces enough grapes for one cask of wine. After consulting a sommelier she find out that she can produces two types of sherry wine:

   - Dry Fino, made from 100% palomino grapes, making a profit of €1000 a cask,
   - Dry Cream, made from 70% palomino grapes and 30% moscatel grapes, making a profit of €900 a cask.

   She wishes to know how many casks (not necessarily integer) of each of the two types of wine she should produce in order to maximise profit.

   (a) Formulate this as a linear programming problem. Clearly define the decision variables, objective function, and all constraints. **[5]**

   (b) Solve the problem using the graphical method:

      i. Draw the feasible region, clearly labelling all constraints. **[5]**

      ii. Evaluate the objective function at each basic feasible solution. **[4]**

   A neighbouring vintner suggests that she could make more money by producing some sweeter wines. She finds two recipes:

   - Sweet Cream, made from 40% palomino grapes and 60% moscatel grapes, making a profit of €900 a cask,
   - Sweet Moscatel, made from 100% moscatel grapes, making a profit of €2000 a cask.

   Sweet Moscatel wine, although profitable, is known not to sell well, and so Señora Martinez wants to produce at most 10 casks of this. She wishes to know how many casks of each of the four types of wine she should produce to in order to maximise profit.

   (c) Formulate this as a linear programming problem. **[5]**

   (d) Why can't we use the graphical method now to solve the problem? **[1]**

   (e) Write out the initial Simplex tableau for this problem. Note that you are not required to complete the Simplex algorithm. **[4]**

   (f) Give an example of computer software that is used to solve linear programming problems. **[1]**

10. Consider the project below with seven activities:

Activity	Duration	Prerequisites	Crash Time	Crash Cost
A	10	-	9	£100
B	15	A	10	£25
C	9	A	7	£40
D	7	C	6	£10
E	11	B, C	10	£12
F	11	D	8	£35
G	5	E, F	5	£5

(a) Draw the activities-on-nodes diagram. Do a forward and backward pass, writing down each activity's float. **[11]**

(b) Write down the critical path. **[1]**

(c) What is the minimum amount of time we can complete the project in? **[1]**

(d) Draw the activities-on-arrows diagram. Do a forward and backward pass. Find the critical path. **[5]**

(e) Find the least expensive way to crash the project so that its duration is 40 time units. **[7]**