Mock Exam 1 - Solutions

Section A

1) Passengers arrive at an airport check-in counter randomly, following a Poisson process with a mean rate of 15 per hour.

a) Find the probability of having at least 3 arrivals in an interval of 4 minutes.

[5]

Let X be the number of arrivals in 4 minutes. Then $X \sim \text{Poisson}(\lambda)$ where $\lambda = 1$.

Now
$$\begin{aligned}
\mathbb{P}(X \geq 3) &= 1 - \mathbb{P}(X < 3) \\
&= 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1) - \mathbb{P}(X=2) \\
&= 1 - \left(\frac{1^0 e^{-1}}{0!} + \frac{1^1 e^{-1}}{1!} + \frac{1^2 e^{-1}}{2!} \right) \\
&= 1 - e^{-1} - e^{-1} - \frac{e^{-1}}{2} \\
&= \underline{0.0803}
\end{aligned}$$

b) What is the mean time between consecutive arrivals?

[2]

Let T be the time between consecutive arrivals. Then $T \sim \text{Expon}(15)$, and
$$\begin{aligned}
\mathbb{E}[T] &= 1/15 \text{ hours} \\
&= \underline{4 \text{ minutes}}
\end{aligned}$$

c) What is the probability that there is at least a 40 second gap between two consecutive arrivals? [3]

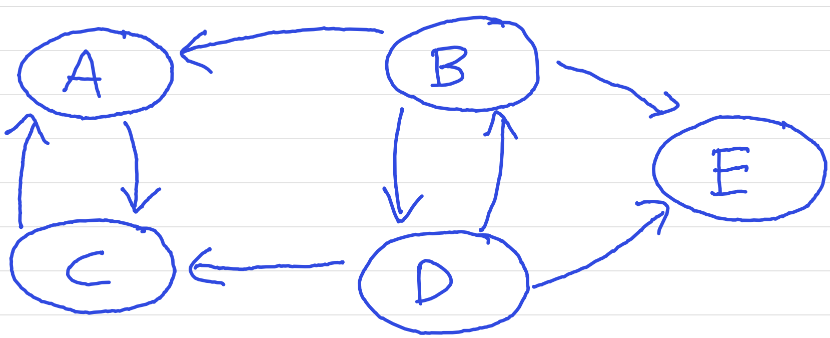
$T \sim \text{Expon}(15 \text{ per hour})$ is equivalent to

$T \sim \text{Expon}(1/4 \text{ per minute})$

$$\mathbb{P}(T \geq 40s) = \mathbb{P}(T \geq \tfrac{2}{3} \text{ mins})$$
$$= 1 - \mathbb{P}(T \leq \tfrac{2}{3} \text{ mins})$$
$$= 1 - (1 - e^{-\frac{1}{4}\frac{2}{3}})$$
$$= e^{-1/6} = 0.8464$$

$$X_1 = 10/3, \quad x_2 = 0$$

2) a) Consider the continuous-time Markov chain on five states visualised below, where transitions are drawn only if there is a non-zero transition rate between two states



i) Write down all its irreducible classes, stating whether they are closed or not closed. [3]

- $\{A, C\}$ closed
- $\{B, D\}$ not closed
- $\{E\}$ closed

ii) Label each state as either Recurrent, Transient, or Absorbing. [5]

A – recurrent
B – transient
C – recurrent
D – transient
E – absorbing

b) Consider the discrete-time Markov chain on three states defined by the transition probability matrix

$$P = \begin{pmatrix} 0.0 & 0.5 & x \\ 0.1 & 0.9 & 0.0 \\ 0.4 & 0.0 & 0.6 \end{pmatrix}$$

i) Give the value x that ensures that P is a valid transition probability matrix for a Markov chain. [2]

$x = 0.5$ as rows should sum to 1.

ii) Find the steady-state probability vector for this Markov chain.

[7]

We want $\underline{\pi}$ such that $\underline{\pi} P = \underline{\pi}$, and $\sum \underline{\pi} = 1$.

So:

$$(\pi_1, \pi_2, \pi_3)\begin{pmatrix} 0.0 & 0.5 & 0.5 \\ 0.1 & 0.9 & 0.0 \\ 0.4 & 0.0 & 0.6 \end{pmatrix} = (\pi_1, \pi_2, \pi_3)$$

$$\therefore \quad \begin{aligned} 0.1\pi_2 + 0.4\pi_3 &= \pi_1 \quad \cdots\cdots ① \\ 0.5\pi_1 + 0.9\pi_2 &= \pi_2 \quad \cdots\cdots ② \\ 0.5\pi_1 + 0.6\pi_3 &= \pi_3 \quad \cdots\cdots ③ \\ \pi_1 + \pi_2 + \pi_3 &= 1 \quad \cdots\cdots ④ \end{aligned}$$

from ③: $0.5\pi_1 = 0.4\pi_3$

$$\pi_3 = \frac{5}{4}\pi_1$$

from ②: $0.5\pi_1 = 0.1\pi_2$

$$\pi_2 = 5\pi_1$$

from ④: $\pi_1 + \pi_2 + \pi_3 = 1$

$$\pi_1\left(1 + 5 + 5/4\right) = 1$$

$$\pi_1 = \frac{4}{29}$$

$$\therefore \quad \underline{\pi} = \left(\frac{4}{29}, \frac{20}{29}, \frac{5}{29}\right)$$

3) At an automated car wash, cars arrive randomly, following a Poisson process with rate $\lambda = 12$ per hour. There are three machines that can wash cars in parallel, and each machine takes exactly 10 minutes to wash a car. If all three machines are busy, then cars queue up to use the machines, and there is room for 6 cars to wait at any one time, if there are 6 cars queueing then arriving cars are turned away and look elsewhere for a car wash. Waiting cars are called to the car washing machines in the order in which they arrived.

a) Using Kendall's notation, describe the system as a queue. [5]

$$M / D / 3 / 9 / FIFO$$

The car wash decides to expand the waiting area, and there is now so much queueing capacity that it can be modelled as infinite.

b) What is the traffic intensity of the system? [1]

$$\rho = \frac{\lambda}{c\mu} = \frac{12}{3 \times 12} = 1/3$$

c) Is this a stable system? [1]

Yes, as $\rho < 1$.

d) On average there are 1·02 cars in the system, what is the average amount of time spent in the system? [2]

From Little's law: $W = \frac{1}{\lambda} L = \frac{1}{12} \times 1{\cdot}02 = \underline{0{\cdot}085}$

4) Use the Simplex algorithm to solve the following linear programming problem:

Minimise $3x_1 - x_2 - 5x_3$

subject to:

$$\begin{aligned} -x_1 + 5x_2 + x_3 &\le 2 \\ 5x_1 + x_2 &\le 7 \\ -2x_2 + x_3 &\le 4 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

[8]

	x_1	x_2	x_3	s_1	s_2	s_3
2	−1	5	1	1	0	0
7	5	1	0	0	1	0
4	0	−2	1	0	0	1
0	3	−1	−5	0	0	0

$\tilde{r}_1 \to r_1$ $\quad \tilde{r}_2 \to r_2$ $\quad \tilde{r}_3 \to r_3 - r_1$ $\quad \tilde{r}_4 \to r_4 + 5r_1$

	x_1	x_2	x_3	s_1	s_2	s_3
2	−1	5	1	1	0	0
7	5	1	0	0	1	0
2	1	−7	0	−1	0	1
10	−2	24	0	5	0	0

$\tilde{r}_2 \to \frac{1}{5} r_2$ $\quad r_1 \to r_1 + \tilde{r}_2$ $\quad r_3 \to r_3 - \tilde{r}_2$ $\quad r_4 \to r_4 + 2\tilde{r}_2$

	X_1	X_2	X_3	S_1	S_2	S_3
$17/5$	0	$26/5$	1	1	$1/5$	0
$7/5$	1	$1/5$	0	0	$1/5$	0
$3/5$	0	$-36/5$	0	-1	$-1/5$	1
$64/5$	0	$122/5$	0	5	$2/5$	0

$\therefore \quad X_1 = 7/5 \qquad X_2 = 0 \qquad X_3 = 17/5$

5) The following tableau is obtained by carrying out the Simplex algorithm:

	X_1	X_2	S_1	S_2	S_3
$10/3$	0	8	1	$-2/3$	0
$10/3$	1	-2	0	$1/3$	0
$8/3$	0	3	0	$-1/3$	1
$10/3$	0	0	0	$1/3$	0

a) Read off a solution given by this tableau. [1]

$X_1 = 10/3 \qquad X_2 = 0$

b) Pivot one more time to find another optimal solution. [4]

	X_1	X_2	S_1	S_2	S_3
$10/3$	0	8	1	$-2/3$	0
$10/3$	1	-2	0	$1/3$	0
$8/3$	0	3	0	$-1/3$	1
$10/3$	0	$0\uparrow$	0	$1/3$	0

$$\tilde{r}_1 \to \frac{1}{8} r_1 \qquad \tilde{r}_2 \to r_2 + 2\tilde{r}_1 \qquad \tilde{r}_3 \to r_3 - 3\tilde{r}_1$$

	X_1	X_2	S_1	S_2	S_3
$5/12$	0	1	$1/8$	$-1/12$	0
$25/6$	1	0	$1/4$	$1/6$	0
$17/12$	0	0	$-3/8$	$-1/12$	1
$10/3$	0	0	0	$1/3$	0

$$X_1 = \frac{25}{6}, \quad X_2 = \frac{5}{12}$$

c) Write down the set of <u>all</u> optimal solutions as a parametrisation of a straight line segment. [2]

$$S = \left\{ \left(\frac{10}{3}, 0\right)(1-t) + t\left(\frac{25}{6}, \frac{5}{12}\right) \ \forall\, t \in [0,1] \right\}$$

$$= \left\{ \left(\frac{10}{3} + \frac{5}{6}t, \frac{5}{12}t\right) \ \forall\, t \in [0,1] \right\}$$

d) If the value of X_2 is fixed as $X_2 = 1/12$, what value must X_1 take for the solution to remain optimal? [3]

$$X_2 = \frac{5}{12}t = \frac{1}{12} \Rightarrow t = 1/5$$

$$X_1 = \frac{10}{3} + \frac{5}{6}t = \frac{10}{3} + \frac{5}{6}\left(\frac{1}{5}\right)$$

$$= 7/2$$

6) A large community club has 2,200 members, consisting of 1,150 children, 450 adults, and 600 old age pensioners. It is organising an annual theatre trip for its members. There are four plays:

- "Romeo & Juliet", with 730 ticket available,
- "As You Like It", with 60 tickets available,
- "King Lear", with 410 tickets available, and
- "The Merchant of Venice" with 1,000 tickets available.

The price of each play is different, and there are different prices for children, adults, and old age pensioners, as shown in the table below:

	Children	Adults	Pensioners
Romeo + Juliet	£2	£5	£3
As You Like It	£11	£15	£10
King Lear	£7	£12	£4
Merchant of Venice	£8	£9	£6

Each member must be bought a ticket to one of the plays.

a) Find a feasible basic allocation of tickets using the minimum cost method. [4]

	C	A	P	
R	730 2	5	3	730
A	11	60 15	10	60
K	7	12	410 4	410
M	420 8	390 9	190 6	1000
	1150	450	600	

b) Use the Stepping-Stone algorithm to find an allocation that minimises the total cost of the tickets. [6]

RA : $5-9+8-2 = 2$
RP : $3-6+8-2 = 3$
AC : $11-15+9-8 = -3$
AP : $10-6+9-15 = -2$
KC : $7-4+6-8 = 1$
KA : $12-4+6-9 = 5$

$\therefore$ Increase AC by 60.

	C	A	P	
R	730 2	5	3	730
A	60 11	15	10	60
K	7	12	410 4	410
M	360 8	450 9	190 6	1000
	1150	450	600	

RA: $5 - 9 + 8 - 2 = 2$
RP: $3 - 6 + 8 - 2 = 3$
AA: $15 - 9 + 8 - 11 = 3$
AP: $10 - 6 + 8 - 11 = 1$
KC: $7 - 4 + 6 - 8 = 1$
KA: $12 - 4 + 6 - 9 = 5$

$\therefore$ We are done, an optimal solution has been found.

c) The theatre showing "As You Like It" want to sell the group more adult tickets, and decide to put on a promotion. What range of values should they sell adult tickets for in order to change the group's optimal allocation? [3]

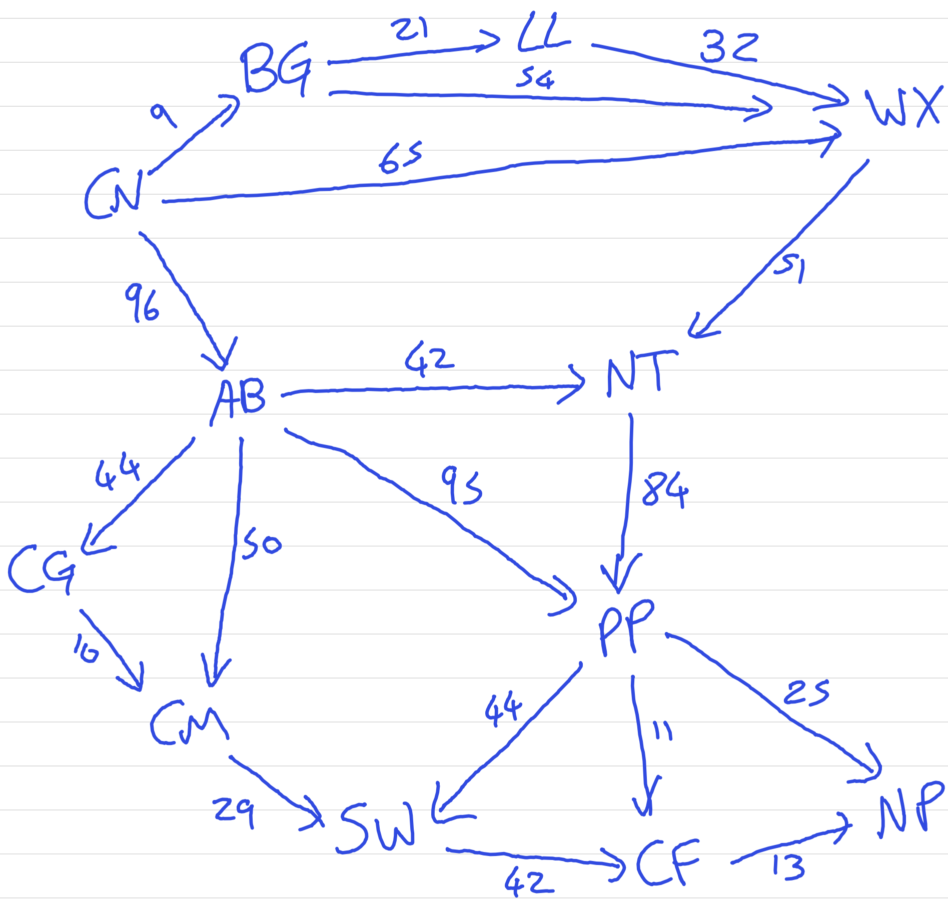
If price of AA is P, then loops through AA are:

$$AA: P - 9 + 8 - 11 = P - 12$$

this should be negative to change the optimal solution.

$$P - 12 < 0 \quad \Rightarrow \quad \underline{P < 12}$$

7) A student is stranden in Caernarfon (CN), and is trying to return home to Newport (NP) via public transport, however there is no direct bus route between the two towns. They have identified the following 19 intermediate bus routes, requiring changes in either Bangor (BG), Llandudno (LL), Wrexham (WX), Aberystwyth (AB), Newtown (NT), Cardigan (CG), Carmarthen (CM), Pontypridd (PP), Swanse (SW) or Cardiff (CF). The routes and their distances are given on the map below:



Use the value iteration algorithm to identify the shortest bus journey from Caernarfon (CN) to Newport (NP). [8]

A natural ordering would be:

NP, CF, SW, PP, CM, CG, NT, AB, WX, LL, BG, CN

Performing value iteration:

V	u	r_{uv}	$r_{uv} + f_u$	f_v
NP	—	—	—	0
CF	NP	13	13+0=13	13
SW	CF	42	42+13=55	55
PP	NP	25	25+0=25	
	CF	11	11+13=24	24
	SW	44	44+55=99	
CM	SW	29	29+55=84	84
CG	CM	10	10+84=94	94
NT	PP	84	84+24=108	108
AB	CG	44	44+94=138	
	CM	50	50+84=134	
	NT	42	42+108=150	119
	PP	95	95+24=119	
WX	NT	51	51+108=159	159
LL	WX	32	32+159=191	191
BG	WX	54	54+159=213	212
	LL	21	21+191=212	
CN	BG	9	9+212=221	
	WX	65	65+159=224	215
	AB	96	96+119=215	

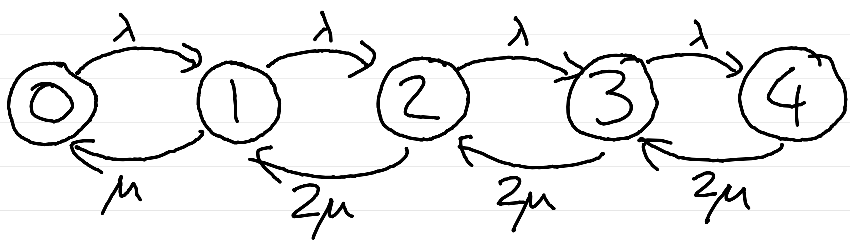
Working backwards the solution can be read as:

CN → AB → PP → CF → NP

Section B

8) a) Consider an M/M/2/4/FIFO queue, with arrival rate λ and service rate μ. This can be modelled as a continuous-time Markov chain on five states representing the number of customers present in the system.

i) Draw a visualisation of this Markov chain. [3]



ii) Give the transition rate matrix Q. [2]

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 & 0 \\ 0 & 2\mu & -(\lambda+2\mu) & \lambda & 0 \\ 0 & 0 & 2\mu & -(\lambda+2\mu) & \lambda \\ 0 & 0 & 0 & 2\mu & -2\mu \end{bmatrix}$$

iii) Find the steady-state probabilities $P_0, P_1, \ldots, P_4$ in terms of λ and μ.

[8]

Steady-state when $\underline{\pi} Q = \underline{0}$, and $\Sigma \pi = 1$.

The system of equations to solve is:

$$-\lambda P_0 + \mu P_1 = 0 \quad \cdots\cdots\cdots \quad (1)$$

$$\lambda P_0 - (\lambda+\mu) P_1 + 2\mu P_2 = 0 \quad \cdots\cdots\cdots \quad (2)$$

$$\lambda P_1 - (\lambda+2\mu) P_2 + 2\mu P_3 = 0 \quad \cdots\cdots\cdots \quad (3)$$

$$\lambda P_2 - (\lambda+2\mu) P_3 + 2\mu P_4 = 0 \quad \cdots\cdots\cdots \quad (4)$$

$$\lambda P_3 - 2\mu P_4 = 0 \quad \cdots\cdots\cdots \quad (5)$$

$$P_0 + P_1 + P_2 + P_3 + P_4 = 1 \quad \cdots\cdots\cdots \quad (6)$$

Putting everything in terms of P_0:

- From (1): $P_1 = \frac{\lambda}{\mu} P_0$

- from (2):

$$\lambda P_0 + 2\mu P_2 = (\lambda+\mu) P_1$$

$$2\mu P_2 = (\lambda+\mu)\frac{\lambda}{\mu} P_0 - \lambda P_0$$

$$2\mu P_2 = \frac{\lambda^2}{\mu} P_0$$

$$P_2 = \frac{\lambda^2}{2\mu^2} P_0$$

- from (3):

$$\lambda P_1 + 2\mu P_3 = (\lambda+2\mu) P_2$$

$$2\mu P_3 = (\lambda+2\mu)\frac{\lambda^2}{2\mu^2} P_0 - \frac{\lambda^2}{\mu} P_0$$

$$P_3 = \frac{\lambda^3}{4\mu^3} P_0$$

- from 5: $2\mu P_4 = \lambda P_3$

$$2\mu P_4 = \lambda \frac{\lambda^3}{4\mu^3} P_0$$

$$P_4 = \frac{\lambda^4}{8\mu^4} P_0$$

- And from 6:

$$P_0 + P_1 + P_2 + P_3 + P_4 = 1$$

$$P_0 \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{4\mu^3} + \frac{\lambda^4}{8\mu^3}\right) = 1$$

$$\therefore P_0 = \frac{1}{1+\frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{4\mu^3} + \frac{\lambda^4}{8\mu^4}}$$

and $P_1 = \frac{\lambda}{\mu} P_0$, $P_2 = \frac{\lambda^2}{2\mu^2} P_0$, $P_3 = \frac{\lambda^3}{4\mu^3}$, $P_4 = \frac{\lambda^4}{8\mu^4}$

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iv) Now using $\lambda = 5$ and $\mu = 5$, find the average number of customers in the system. [3]

$$P_0 = \frac{8}{23}, \quad P_1 = \frac{8}{23}, \quad P_2 = \frac{4}{23}, \quad P_3 = \frac{2}{23}, \quad P_4 = \frac{1}{23}$$

$$L = \sum_{k=0}^{4} k P_k = \left(0 \times \frac{8}{23}\right) + \left(1 \times \frac{8}{23}\right) + \left(2 \times \frac{4}{23}\right) + \left(3 \times \frac{2}{23}\right) + \left(4 \times \frac{1}{23}\right)$$

$$= 26/23$$

b) Consider an M/M/1 queue with arrival rate λ, service rate μ, and traffic intensity $\rho = \lambda/\mu$. The steady-state probabilities are given by

$$P_k = \rho^k P_0$$

$$P_0 = 1 - \rho$$

i) What is the probability that an arriving customer has to wait?

[1]

An arriving customer waits if there is more than 0 customers waiting:

$$\mathbb{P}(\text{arriving customer waits}) = 1 - P_0$$
$$= \rho.$$

ii) Using the above, show that the expected number of customers in the system is given by

$$L = \frac{\rho}{1-\rho}$$

[4]

$$L = \sum_{k=0}^{\infty} k P_k$$

$$= \sum_{k=0}^{\infty} k \rho^k P_0 = \sum_{k=0}^{\infty} k \rho^k (1-\rho)$$

$$= (1-\rho) \sum_{k=0}^{\infty} k \rho^k = (1-\rho) \frac{\rho}{(1-\rho)^2}$$

$$= \frac{\rho}{1-\rho}$$

iii) Using Little's law, show that the expected time spent in the system is given by

$$W = \frac{1}{\mu-\lambda}$$

[4]

$$W = \frac{1}{\lambda}L$$

$$= \frac{\rho}{\lambda(1-\rho)} = \frac{1/\mu}{1-\rho}$$

$$= \frac{1}{\mu-\mu\rho} = \underline{\frac{1}{\mu-\lambda}}$$

9) Señora Martinez has inherited a number of vineyards in the Sherry Triangle in Spain. She now has 500 acres growing palomino grapes, and 100 acres growing moscatel grapes. Each acre of vineyard produces enough grapes for one cask of wine. After consulting a sommelier she finds out that she can produce two types of sherry wine:

- Dry fino, made from 100% palomino grapes, making a profit of €1000 a cask,
- Dry cream, made from 70% palomino grapes and 30% moscatel grapes, making a profit of €900 a cask.

She wishes to know how many casks (not necessarily integer) of each of the two types of wine she should produce in order to maximise profit.

a) Formulate this as a linear programming problem. Clearly define the decision variables, objective function, and all constraints. [5]

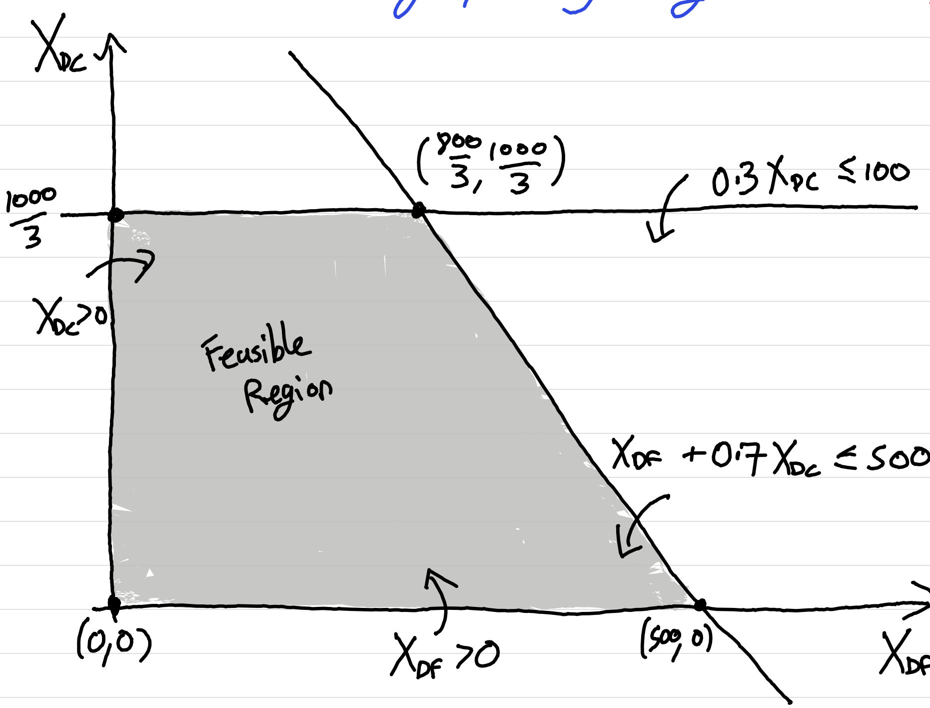
Let X_{DF} be the number of casks of dry fino,
Let X_{DC} be the number of casks of dry cream.

maximise: $1000 X_{DF} + 900 X_{CD}$

subject to:
$$X_{DF} + 0.7 X_{DC} \leq 500$$
$$0.3 X_{DC} \leq 100$$
$$X_{DF}, X_{DC} \geqslant 0$$

b) Solve the problem using the graphical method:

i) Draw the feasible region, clearly labelling all constraints. [5]



ii) Evaluate the objective function at each basic feasible solution. [4]

Solution (X_{DF}, X_{DC})	Objective $1000X_{DF}+900X_{DC}$
$(0,0)$	€0
$(0,\frac{1000}{3})$	€300,000
$(\frac{800}{3},\frac{1000}{3})$	€566,666.66
$(500,0)$	€500,000

$\therefore$ The optimal solution is $(\frac{800}{3}, \frac{1000}{3})$.

A neighbouring vintner suggests that she could make more money by producing some sweeter wines. She finds two recipes:

- Sweet Cream, made from 40% palomino grapes and 60% moscatel grapes, making a profit of €900 a cask,
- Sweet Moscatel, made from 100% moscatel grapes, making a profit of €2000 a cask.

Sweet moscatel wine, although profitable, is not known to sell well, and so Señora Martinez wants to produce at most 10 casks of this. She wishes to know how many casks of each of the four types of wine she should produce in order to maximise profit.

c) Formulate this as a linear programming problem. [5]

Let X_{DF} be the number of casks of Dry fino,
Let X_{DC} be the number of casks of Dry Cream,
Let X_{SC} be the number of casks of Sweet Cream,
Let X_{SM} be the number of casks of Sweet Moscatel.

Maximise: $1000X_{DF} + 900X_{DC} + 900X_{SC} + 2000X_{SM}$

subject to:

$$
\begin{aligned}
X_{DF} + 0.7X_{DC} + 0.4X_{SC} &\leq 500 \\
0.3X_{DC} + 0.6X_{SC} + X_{SM} &\leq 100 \\
X_{SM} &\leq 10 \\
X_{DF}, X_{DC}, X_{SC}, X_{SM} &\geq 0
\end{aligned}
$$

d) Why can't we ose the graphical method now to solve the problem? [1]

Now there are 4 desicion variables, and the graphical method can only be osed when there are two.

e) Write out the initial Simplex tableau for this problem. Note that you are not required to complete the Simplex algorithm. [4]

	X_{DF}	X_{DC}	X_{SC}	X_{SM}	S_1	S_2	S_3
500	1	7/10	4/10	0	1	0	0
100	0	3/10	6/10	0	0	1	0
10	0	0	0	1	0	0	1
0	−1000	−900	−900	−2000	0	0	0

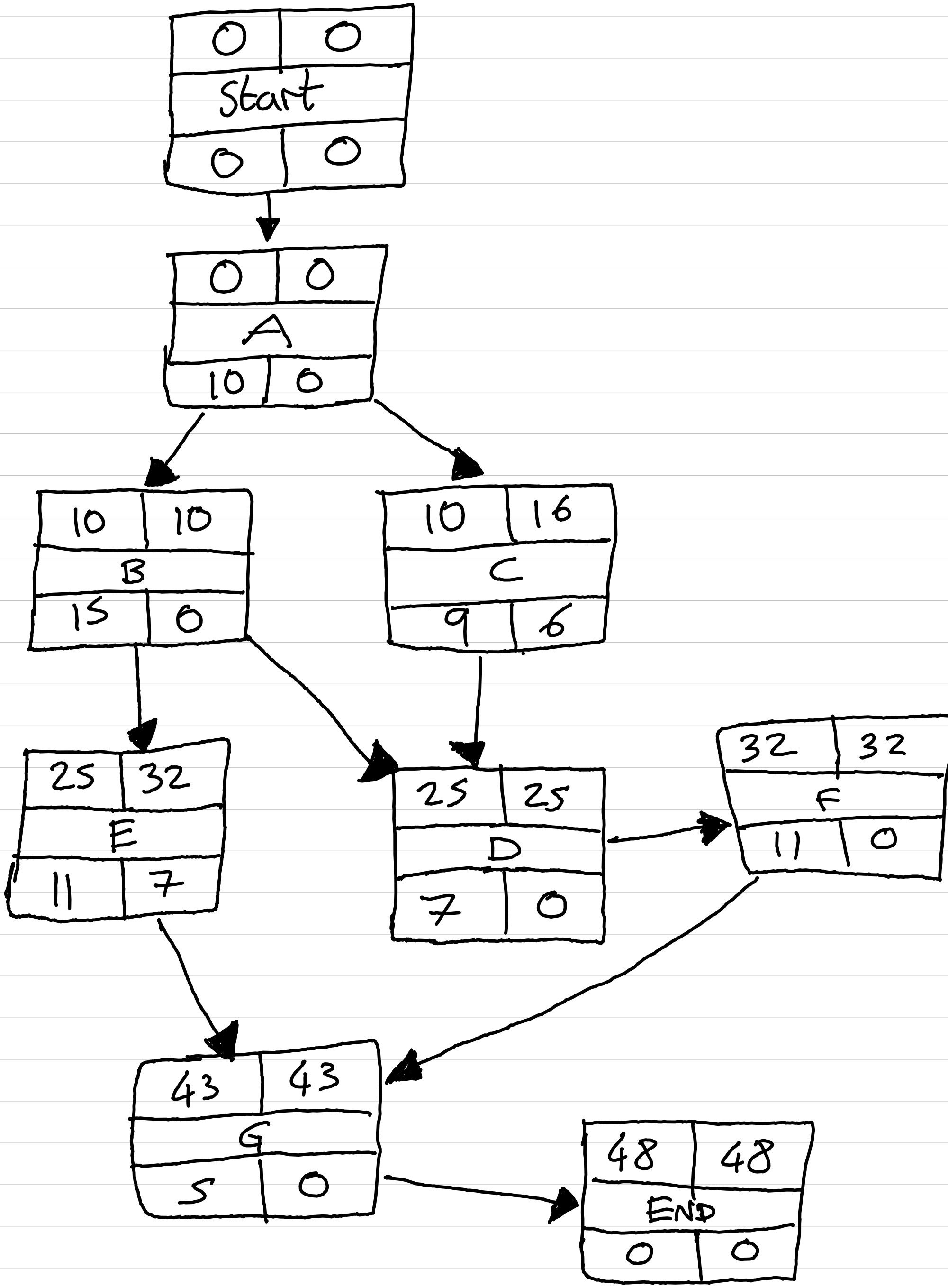
f) Give an example of computer software that is used to solve linear programming problems. [1]

Gurobi / CPLEX / PuLP

10) Consider the project below with seven activities:

Activity	Duration	Prerequisites	Crash Time	Crash Cost
A	10	—	9	£100
B	15	A	10	£25
C	9	A	7	£40
D	7	C	6	£10
E	11	B, C	10	£12
F	11	D	8	£35
G	5	E, F	5	£5

a) Draw the activities-on-nodes diagram. Do a forward and backward pass, writing down each activity's float. [11]


0 0
Start
0 0
0 0
A
10 0
10 10
B
15 0
10 16
C
9 6
25 32
E
11 7
25 25
D
7 0
32 32
F
11 0
43 43
G
5 0
48 48
END
0 0

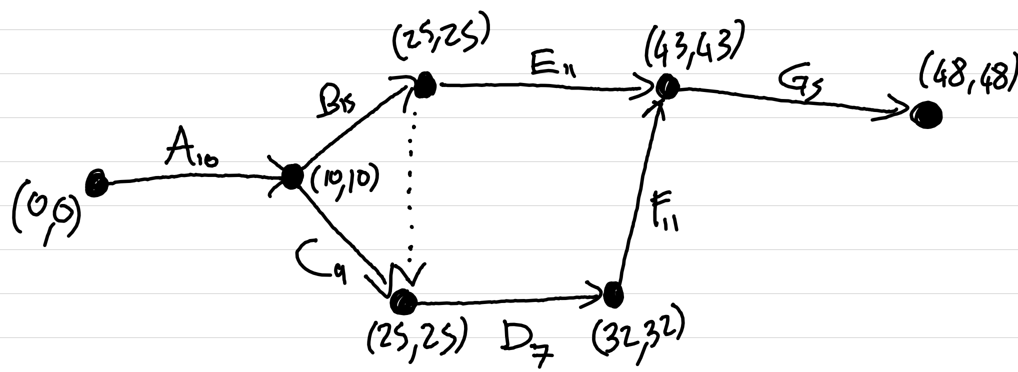
b) Write down the critical path. [1]

A – B – D – F – G

c) What is the minimum amount of time we can complete the project in? [1]

48 time units

d) Draw the activities-on-arrows diagram. Do a forward and backward pass. Find the critical path. [5]



Critical path: A B D F G

e) Find the least expensive way to crash the project so that its duration is 40 time units. [7]

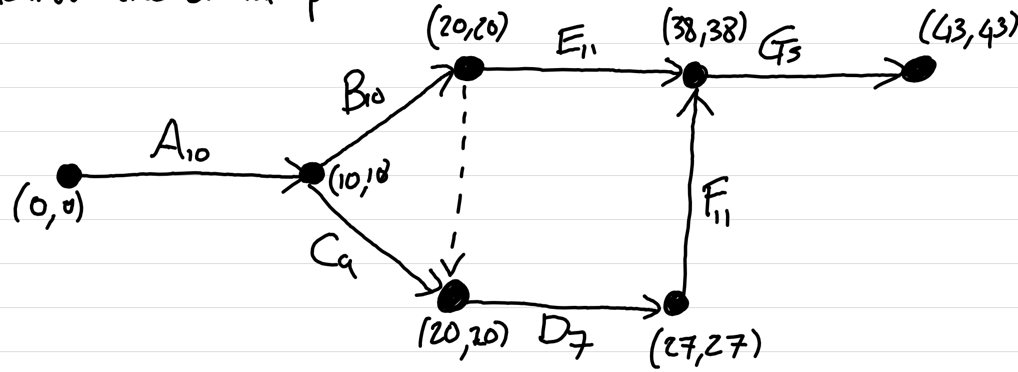
Unit crash costs:

A: 100
B: 5
C: 20
D: 10

E: 12
F: $35/3 = 11.\dot{6}$
G: —

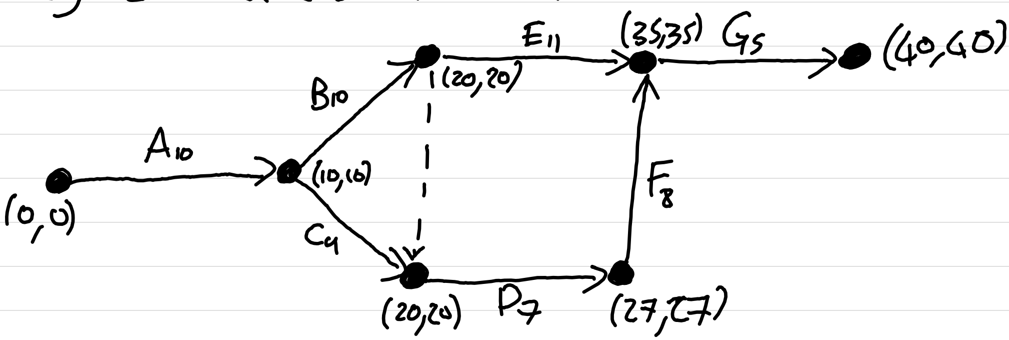
Cheapest to crash is B.
B can be crushed by up to 6 time units before it falls off the critical path. But its crash time is 5. So reduce B by 5, at a cost of $5 \times £5 = £25$.

Redraw the critical path:



Critical path remains ABDFG

B is at crash cost. The next cheapest to crash is F. F can crash by 7 before it falls off the critical path. F can only reduce by 3 before it reaches crash time. So reduce F by 3 at a cost of £35.



Which reduces the project duration to 40 time units, by reducing B by 5, F by 3, at a total cost of £25 + £35 = £60.