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Academic Year: Mock Exam 2

Examination Period: Mock Exam 2

Module Code: MA2601

Examination Paper Title: Operational Research

Duration: 3 hours

Please read the following information carefully:

Structure of Examination Paper:

- There are **7** pages including this page.
- There are **TWO** sections.
- There are **10** questions in total.
- There are no appendices.
- The maximum mark for the examination paper is 150 and the mark obtainable for a question or part of a question is shown in brackets alongside the question.

Instructions for completing the examination:

- Complete the front cover of any answer books used.
- This examination paper must be submitted to an Invigilator at the end of the examination.
- Answer **ALL** questions from **Section A**, and **ALL** questions from **Section B**.
- Each question should be answered on a separate page.

You will be provided with / or allowed:

- **ONE** answer book.
- Squared graph paper.
- The **use of calculators is permitted** in this examination.
- The use of a translation dictionary between English or Welsh and another language, provided that it bears an appropriate departmental stamp, is permitted in this examination.

Section A

(Answer ALL Questions)

1. Arrivals at the NHS 111 service occur at random (Poisson distributed) at a mean rate of 10 per hour.

   (a) What is the probability of observing more than 2 arrivals in an interval of 3 minutes? **[4]**

   (b) What is the probability that there is at least a 36 seconds break between two consecutive arrivals? **[3]**

   (c) Each call, upon arrival, has a 32% chance of being classified as "Urgent" by the triage nurse. What is the probability of observing more than 2 "Urgent" arrivals in the space of 3 minutes? **[4]**

2. Consider a $D/M/1/\infty$/SIRO queue with arrival rate $\lambda = 6$ per hour and service rate $\mu = 10$ per hour.

   (a) Describe the arrival process. **[1]**

   (b) Describe the service time distribution. **[1]**

   (c) Describe the service discipline. **[1]**

   (d) What is the traffic intensity? **[1]**

   (e) Discrete Event Simulation would be used to find the average number of customers in the system. Suggest a peice of software that could be used? **[1]**

   (f) If the average number of customers present in the system is 0.88, what is the average time customers spend in the system? **[2]**

3. The Inverse Distribution Method can be used for generating random numbers.

   (a) Find a function that transforms uniformly distributed random numbers between 0 and 1 into Exponentially distributed random numbers with rate λ. **[3]**

   (b) Use the 5 random numbers provided below to generate Exponentially distributed random numbers with parameter $\lambda = 0.2$: **[3]**

0.123	0.456	0.421	0.796	0.502

4. Consider the discrete-time Markov chain on three states defined by the transition probability matrix

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \\ x & 1/2 & 1/4 \end{pmatrix}$$

(a) What value of x ensures that P is a valid transition probability matrix? **[1]**

(b) The probability of being in each state at time step t is given by π_t.
If $\pi_0 = (1/2, 0, 1/2)$, find π_1 and π_2. **[4]**

(c) Find the steady-state probability vector for this Markov chain. **[7]**

5. Consider the following linear programming problem:

$$\begin{aligned}
&\text{Maximise } 14X_1 + 6X_2 \\
&\text{subject to} \\
&\quad 2X_1 + 3X_2 \geq 6 \\
&\quad -X_1 + X_2 \leq 3 \\
&\quad 7X_1 + 3X_2 \leq 49 \\
&\quad X_1 \leq 6 \\
&\quad X_1, X_2 \geq 0
\end{aligned}$$

(a) Solve the problem using the graphical method:

i. Draw the feasible region, clearly labeling all constraints. **[4]**

ii. Evaluate the objective function at each basic feasible solution. **[3]**

iii. Write down the set of *all* optimal solutions. **[2]**

(b) If we wish to fix $X_2 = 4$, what value should X_1 take for the solution to remain optimal? **[2]**

6. A project has nine tasks that must be completed. Each task takes a fixed amount of time, and may only be completed once any prerequisite tasks have been completed. The tasks are detailed in the table below:

Task	Duration	Prerequisites
A	10	-
B	7	A
C	3	A
D	4	B
E	11	B, C
F	3	D
G	1	E
H	8	F, G
I	8	E

   (a) Draw an activity on arrows diagram and find the critical path. **[5]**

   (b) Draw an activity on nodes diagram and find the critical path. **[7]**

   (c) For each task, state how many time units they can be delayed by before affecting the overall duration of the project. **[2]**

7. A village police station has one sniffer dog that has currently been in service for 6 years. Every three years the police station must decide whether to retire the dog and train a new one, or to retrain its current sniffer dog. Training a new dog costs £30k. Retraining costs increase as the dog's current years of service increases. Similarly, a national animal welfare charity offers grants to the police station to retire the dog early. The dog must retire once it has served the maximum of 9 years of service. Retraining costs and grant amounts are given in the table below.

Current years of service	3	6	9
Retraining cost	£3k	£10k	-
Grant amount	£12k	£7k	£0k

   In 15 years time a new system will be implemented and so the dog will have to be retired. Defining a state as the tuple (y, n), where y is the number of years left in the plan, and n is the number of years the dog has been in service, use dynamic programming to find a plan for the next 15 years:

   (a) Draw the directed acyclic graph. **[6]**

   (b) Perform value iteration on the edges. **[6]**

   (c) Read off the solution and give a plan for the next 15 years. **[2]**

Section B
(Answer ALL questions)

8. Consider an $M/M/1$ queue, with arrival rate λ and service rate μ.

(a) Explain what happens to the queue in the cases where

i. $\lambda > \mu$, and

ii. $\lambda < \mu$. **[2]**

(b) What is the probability that an arriving customer does not have to wait for service? **[2]**

(c) The average number of customers in the system is given by

$$L = \frac{\rho}{1 - \rho}$$

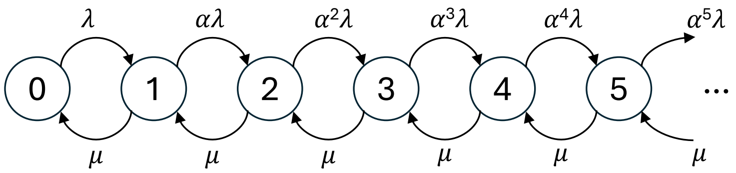
where $\rho = {}^{\lambda}/_{\mu}$. Derive expressions for:

i. W, the average time spent in the system; **[4]**

ii. W_q, the average time spent in the queue; **[2]**

iii. L_q, the average number of customers waiting in the queue. **[4]**

Now consider an $M/M/1$ queue where customers *baulk*, that is choose not to join the queue. Customers baulk with probability $b(n) = \alpha^n$ when there are n customers already in the system upon arrival. The continuous-time Markov chain describing this system is given by:



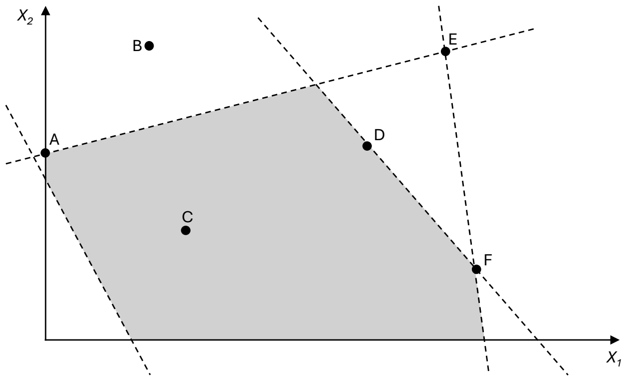
(d) Give an expression for the average holding time of state n, that is the average amount of time the system stays in state n before transitioning to another state. **[3]**

(e) Show by induction that:

$$P_n = \alpha^{T(n-1)} \rho^n P_0$$

for all n, where $T(n) = \sum_{k=0}^{n} k$, the sum of the first n integers. **[8]**

9. (a) Consider the solution space (X_1, X_2) visualised below, with six constraints, including the non-negativity constraints. The feasible region is shaded in gray.



Six solutions are indicated. Categorise them by filling the table below:

	Basic	Non-basic
Feasible		
Infeasible		

[6]

(b) Re-write the following problem as a linear programming problem in standard form. You may need to introduce slack and/or dummy variables. **[6]**

$$
\begin{aligned}
&\text{Maximise } \min(X_1, X_2) \\
&\text{subject to} \\
&\qquad X_1 + X_2 \leq 9 \\
&\qquad 5X_1 - 2X_2 \leq 12 \\
&\qquad X_1, X_2 \geq 0
\end{aligned}
$$

[Note: you are not expected to solve the linear programming problem.]

(c) Solve the following problem using the two-phase method: **[12]**

$$
\begin{aligned}
&\text{Minimise } X_1 + 2X_2 + 3X_3 \\
&\text{subject to} \\
&\qquad 3X_1 + 2X_2 \geq 6 \\
&\qquad 3X_1 + 2X_2 - 2X_3 \leq 1 \\
&\qquad -X_2 + X_3 \leq 8 \\
&\qquad X_1, X_2, X_3 \geq 0
\end{aligned}
$$

(d) Give an example of software used to solve linear programming problems. **[1]**

10. There are three factories that produce caster sugar (A, B, and C), each can produce 9 tonnes of sugar a month. There are three bakeries that order the sugar (X, Y, and Z), demanding 7.3 tonnes, 10.5 tonnes, and 12.1 tonnes of sugar per month, respectively.

The route costs are given in the table below:

	A	B	C
X	2	7	2
Y	4	11	6
Z	7	3	1

We wish to find an allocation that minimises the total transportation cost.

(a) Letting X_{ij} be the amount of goods to transport from factory i to bakery j, formulate the problem as a linear programming problem, writing out the objective function and all constraints. **[6]**

[Note: you are not expected to solve the linear programming problem.]

(b) Use the minimum cost rule to find a feasible basic solution. (Note that the problem is infeasible, so a dummy factory may need to be used.) **[4]**

(c) Use the stepping-stone algorithm to find an optimal allocation. **[5]**

(d) What is the cost of the optimal allocation? **[1]**

(e) Bakery Y is considering charging a fine for each tonne of demand not satisfied. What is the maximum fine they could charge before the solution found in part (c) is no longer optimal? **[4]**

(f) Bakery X decides to boycott factory A over reports of child labor, therefore route XA becomes unavailable. Find a new optimal route. **[5]**