Mock Exam 2 – Solutions

Section A

1) Arrivals at the NHS 111 service occur at random (Poisson distributed) at a mean rate of 10 per hour.

a) What is the probability of observing more than 2 arrivals in an interval of 3 minutes? [4]

Let X be the number of arrivals in 3 minutes, then $X \sim \text{Poisson}(\lambda)$ where $\lambda = 1/2$.

Now
$$\begin{aligned}
\mathbb{P}(X>2) &= 1-\mathbb{P}(X\leq 2) \\
&= 1-\mathbb{P}(X=0)-\mathbb{P}(X=1)-\mathbb{P}(X=2) \\
&= 1-\frac{(1/2)^0 e^{-1/2}}{0!}-\frac{(1/2)^1 e^{-1/2}}{1!}-\frac{(1/2)^2 e^{-1/2}}{2!} \\
&= 1-e^{-1/2}-\tfrac{1}{2}e^{-1/2}-\tfrac{1}{8}e^{-1/2} \\
&= \underline{0.0144}
\end{aligned}$$

b) What is the probability that there is at least a 36 second break between two consecutive arrivals? [3]

Let T be the time between two consecutive arrivals, then

$T \sim \text{Expon}(\lambda)$ where $\lambda = \frac{1}{6}$ per minute
$\lambda = 1/360$ per second.

$$\mathbb{P}(T > 36s) = 1 - \mathbb{P}(T \leq 36s)$$
$$= 1 - \left(1 - e^{-36/360}\right)$$
$$= e^{-1/10}$$
$$= 0.9048$$

c) Each call, upon arrival, has a 32% chance of being classified as "Urgent" by the triage nurse. What is the probability of observing more than 2 "Urgent" arrivals in the space of 3 minutes? [4]

We have X, the number of arrivals in 3 minutes, with $X \sim \text{Poisson}(1/2)$.

Let X_u be the number of "Urgent" arrivals in 3 minutes, then:

$$X_u \sim \text{Poisson}\left(0.32 \times \tfrac{1}{2}\right)$$
$$X_u \sim \text{Poisson}(0.16)$$

So
$$\mathbb{P}(X_u > 2) = 1 - \mathbb{P}(X_u \leq 2)$$
$$= 1 - \mathbb{P}(X_u = 0) - \mathbb{P}(X_u = 1) - \mathbb{P}(X_u = 2)$$
$$= 1 - \frac{(0.16)^0 e^{-0.16}}{0!} - \frac{(0.16)^1 e^{-0.16}}{1!} - \frac{(0.16)^2 e^{-0.16}}{2!}$$
$$= 6.058 \times 10^{-4}$$

2) Consider a D/M/1/∞/SIRO queue with arrival rate $\lambda = 6$ per hour and service rate $\mu = 10$ per hour.

a) Describe the arrival process. [1]

Deterministic arrivals every 10 minutes.

b) Describe the service time distribution. [1]

Exponentially distributed with rate 10 per hour

c) Describe the service discipline. [1]

Service in Random Order

d) What is the traffic intensity? [1]

$$\rho = \frac{\lambda}{c\mu} = \frac{6}{1 \times 10} = \frac{3}{5}$$

e) Discrete event Simulation would be used to find the average number of customers in the system. Suggest a piece of software that could be used. [1]

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f) If the average number of customers present in the system is 0.88, what is the average time spent in the system? [2]

$$W = \frac{1}{\lambda} L$$

$$= \frac{1}{6} 0.88$$

$$= 0.14\dot{6} \text{ hours} = 8.8 \text{ minutes}$$

3) The Inverse Distribution Method can be used for generating random numbers.

a) Find a function that transforms uniformly distributed random numbers between 0 and 1 into Exponentially distributed random numbers with rate λ. [3]

The CDF of the Exponential distribution is:

$$F(x) = 1 - e^{-\lambda x}$$

then the required function is $F^{-1}(x)$,

$$y = 1 - e^{-\lambda x}$$

$$e^{-\lambda x} = 1 - y$$

$$-\lambda x = \ln(1-y)$$

$$x = -\frac{1}{\lambda}\ln(1-y)$$

$$x = \frac{1}{\lambda}\ln\left(\frac{1}{1-y}\right)$$

$$\therefore \quad F^{-1}(x) = \frac{1}{\lambda}\ln\left(\frac{1}{1-x}\right)$$

b) Use the 5 random numbers provided below to generate Exponentially distributed random numbers with parameter $\lambda = 0.2$:

0.123	0.456	0.421	0.796	0.502

[3]

$F^{-1}(0.123) = 0.262$

$F^{-1}(0.456) = 1.218$

$F^{-1}(0.421) = 1.093$

$F^{-1}(0.796) = 3.179$

$F^{-1}(0.502) = 1.394$

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4) Consider the discrete-time Markov chain on three states defined by the transition probability matrix

$$P = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \\ x & 1/2 & 1/4 \end{bmatrix}$$

a) What value of x ensures that P is a valid transition probability matrix? [1]

$x = 1/4$ as rows should sum to 1.

b) The probability of being in each state at time step t is given by π_t. If $\pi_0 = (1/2, 0, 1/2)$, find π_1 and π_2. [4]

$$\pi_1 = \pi_0 P$$
$$= (1/2, 0, 1/2)\begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$
$$= \left(\frac{1}{8} + 0 + \frac{1}{8},\ \frac{1}{8} + 0 + \frac{1}{4},\ \frac{1}{4} + 0 + \frac{1}{8}\right)$$
$$= \underline{\left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)}$$

$$\pi_2 = \pi_1 P$$
$$= \left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)\begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$
$$= \left(\frac{1}{16} + 0 + \frac{3}{32},\ \frac{1}{16} + \frac{3}{16} + \frac{3}{16},\ \frac{1}{8} + \frac{3}{16} + \frac{3}{32}\right)$$
$$= \underline{\left(\frac{5}{32}, \frac{7}{16}, \frac{13}{32}\right)}$$

c) Find the steady-state probability vector for this Markov chain. [7]

We need to find π such that $\pi P = \pi$ and $\Sigma \pi = 1$, that is, solve:

$$\frac{1}{4}\pi_1 + \frac{1}{4}\pi_3 = \pi_1 \quad \cdots\cdots (1)$$

$$\frac{1}{4}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = \pi_2 \quad \cdots\cdots (2)$$

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = \pi_3 \quad \cdots\cdots (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \quad \cdots\cdots (4)$$

from (1):

$$\frac{1}{4}\pi_1 + \frac{1}{4}\pi_3 = \pi_1$$

$$\frac{1}{4}\pi_3 = \frac{3}{4}\pi_1$$

$$\underline{\pi_3 = 3\pi_1}$$

from (3):

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{1}{4}\pi_3 = \pi_3$$

$$\frac{1}{2}\pi_1 + \frac{1}{2}\pi_2 + \frac{3}{4}\pi_1 = 3\pi_1$$

$$\frac{1}{2}\pi_2 = \pi_1\left(3 - \frac{3}{4} - \frac{1}{2}\right)$$

$$\pi_2 = 2\frac{7}{4}\pi_1$$

$$\underline{\pi_2 = \frac{7}{2}\pi_1}$$

from (4):

$$\pi_1 + \pi_2 + \pi_3 = 1$$

$$\pi_1 + \frac{7}{2}\pi_1 + 3\pi_1 = 1$$

$$\pi_1\left(1 + \frac{7}{2} + 3\right) = 1$$

$$\pi_1 = \frac{2}{15}$$

$$\therefore \pi_1 = \frac{2}{15}, \quad \pi_2 = \frac{7}{15}, \quad \pi_3 = \frac{6}{15}$$

$$\therefore \underline{\pi = \left(\frac{2}{15}, \frac{7}{15}, \frac{6}{15}\right)}$$

5) Consider the following linear programming problem:

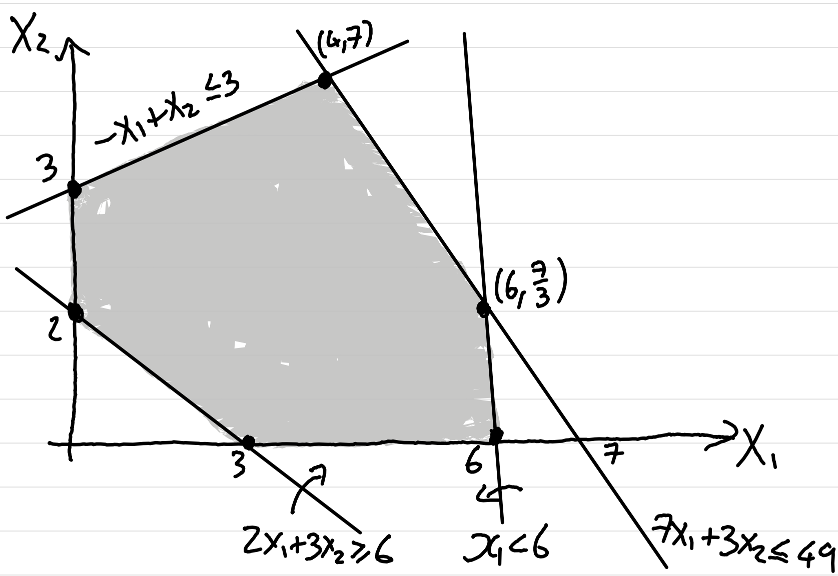
Maximise $14X_1 + 6X_2$

subject to:

$$\begin{aligned} 2X_1 + 3X_2 &\geq 6 \\ -X_1 + X_2 &\leq 3 \\ 7X_1 + 3X_2 &\leq 49 \\ X_1 &\leq 6 \\ X_1, X_2 &\geq 0 \end{aligned}$$

a) Solve the problem using the graphical method:

i) Draw the feasible region, clearly labelling all constraints. [4]



ii) Evaluate the objective function at each basic feasible solution. [3]

Solution (x_1, x_2)	Objective $14x_1 + 6x_2$
$(3,0)$	42
$(0,2)$	12
$(0,3)$	18
$(4,7)$	98
$(6,7/3)$	98
$(6,0)$	84

iii) Write down the set of all optimal solutions. [2]

$$\{(4,7)t + (6,7/3)(1-t) \ \forall\ t \in [0,1]\}$$

$$= \left\{\left(6-2t,\ \frac{7}{3}+\frac{14}{3}t\right) \ \forall\ t \in [0,1]\right\}$$

alternatively:

$$\{(6,7/3)t + (4,7)(1-t) \ \forall\ t \in [0,1]\}$$

$$= \left\{\left(4+2t,\ 7-\frac{14}{3}t\right) \ \forall\ t \in [0,1]\right\}$$

b) If we wish to fix $X_2 = 4$, what value should X_1 take for the solution to remain optimal? [2]

$$X_2 = 7 - \frac{14}{3}t = 4$$

$$7 - 4 = \frac{14}{3}t$$

$$3 = \frac{14}{3}t$$

$$9 = 14t$$

$$t = \frac{9}{14}$$

$$\Rightarrow X_1 = 4 + 2t$$

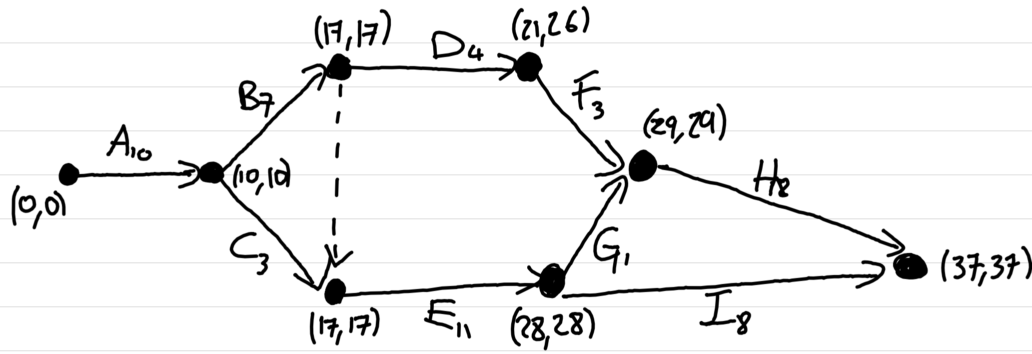
$$= 4 + 2(9/14)$$

$$= 37/7$$

6) A project has nine tasks that must be completed. Each task takes a fixed amount of time, and may only be completed once any prerequisite tasks have been completed. The tasks are detailed in the table below:

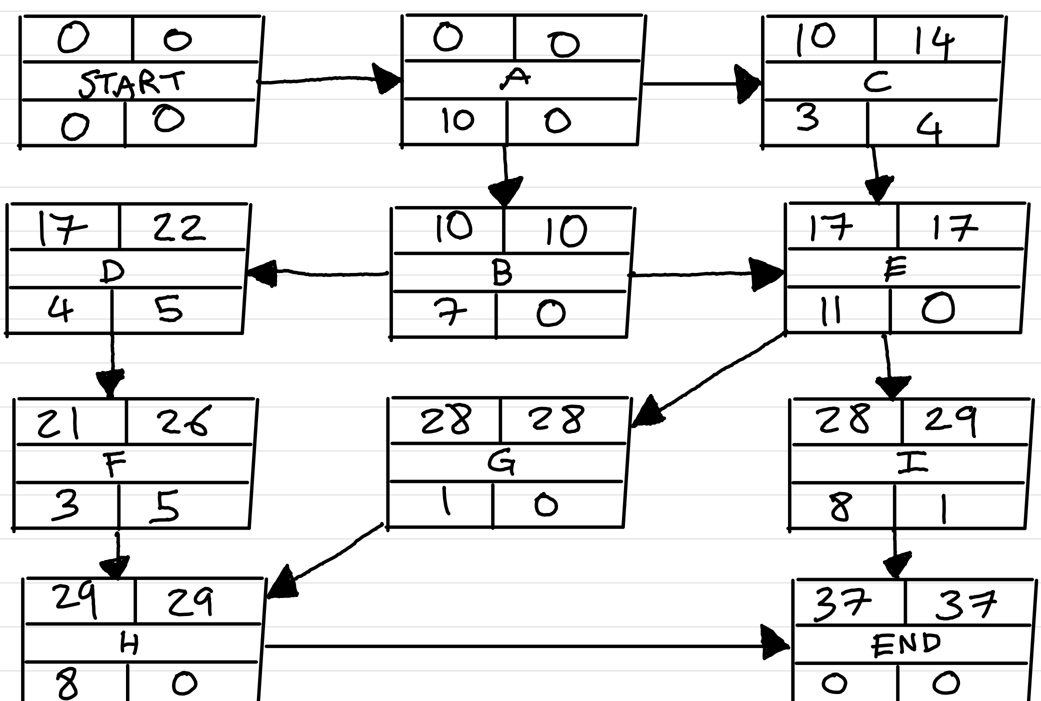
Task	Duration	Prerequisites
A	10	–
B	7	A
C	3	A
D	4	B
E	11	B, C
F	3	D
G	1	E
H	8	F, G
I	8	E

a) Draw an activity on arrows diagram and find the critical path. [5]



Critical path: A-B-E-G-H

b) Draw the activities on nodes diagram and find the critical path. [7]



The critical path is

A - B - E - G - H.

c) For each task, state how many time units they can be delayed by before affecting the overall duration of the project. [2]

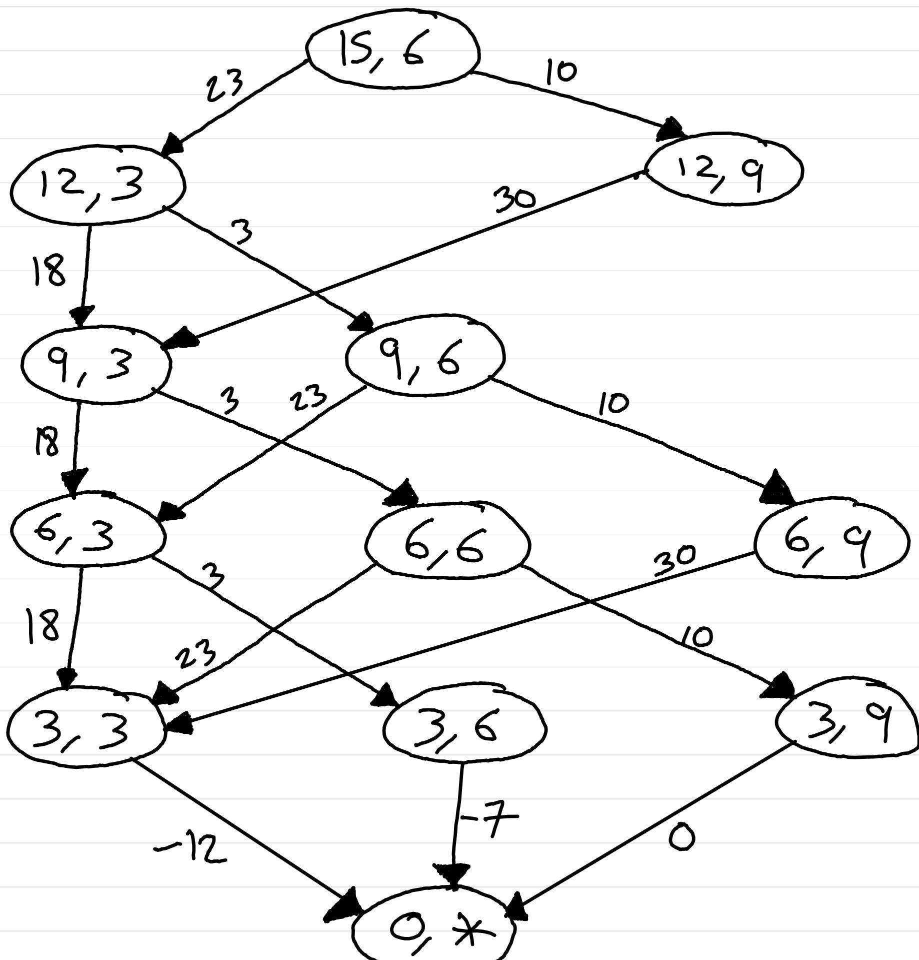
A - 0
B - 0
C - 4
D - 5
E - 0
F - 5
G - 0
H - 0
I - 1

7) A village police station has one sniffer dog that has currently been in service for 6 years. Every three years the police must decide whether to retire the dog and train a new one, or to retrain its current sniffer dog. Training a new dog costs £30k. Retraining costs increase as the dog's current years of service increases. Similarly, a national animal welfare charity offers grants to the police station to retire the dog early. The dog must retire once it has served the maximum of 9 years of service. Retraining costs and grant amounts are given in the table below.

Current years of Service	3	6	9
Retraining Cost	£3k	£10k	—
Grant Amount	£12k	£7k	£0k

In 15 years time a new system will be implemented and so the dog will have to be retired. Defining a state as the tuple (y, n), where y is the number of years left in the plan, and n is the number of years the dog has been in service, use dynamic programming to find a plan for the next 15 years:

a) Draw the directed acyclic graph. [6]



b) Perform value iteration on the edges. [6]

u	v	r_{uv}	$r_{uv}+f_v$	f_u
$(0,*)$	—	—	—	0
$(3,3)$	$(0,*)$	-12	$-12+0$	-12
$(3,6)$	$(0,*)$	-7	$-7+0$	-7
$(3,9)$	$(0,*)$	0	$0+0$	0
$(6,3)$	$(3,3)$ $(3,6)$	18 3	$18-12$ $3-7$	-4
$(6,6)$	$(3,3)$ $(3,9)$	23 10	$23-12$ $10-0$	10
$(6,9)$	$(3,3)$	30	$30-12$	18
$(9,3)$	$(6,3)$ $(6,6)$	18 3	$18-4$ $3+10$	13
$(9,6)$	$(6,3)$ $(6,9)$	23 10	$23-4$ $10+18$	19
$(12,3)$	$(9,3)$ $(9,6)$	18 3	$18+13$ $3+18$	21
$(12,9)$	$(9,3)$	30	$30+13$	43
$(15,6)$	$(12,3)$ $(12,9)$	23 10	$23+21$ $10+43$	44

c) Read off the solution and give a plan for the next 15 years. [2]

$$(15,6)-(12,3)-(9,6)-(6,3)-(3,6)-(0,*)$$

that is replace the dog once it reaches 6 years of service.

Section B

8) Consider an M/M/1 queue, with arrival rate λ and service rate μ.

a) Explain what happens to the queue in the cases where
   i) $\lambda > \mu$, and
   ii) $\lambda < \mu$. [2]

When $\lambda > \mu$ then $\rho = \frac{\lambda}{\mu} > 1$, and the queue grows indefinitely.

When $\lambda < \mu$ then $\rho = \frac{\lambda}{\mu} < 1$, and the queue is stable.

b) What is the probability that an arriving customer does not have to wait for service? [2]

$$P_0 = 1 - \rho = 1 - \frac{\lambda}{\mu}$$

c) The average number of customers in the system is given by

$$L = \frac{\rho}{1-\rho}$$

where $\rho = \lambda/\mu$. Derive expressions for:

i) W, the average time spent in the system. [4]

From Little's law:

$$W = \frac{1}{\lambda} L$$

$$= \frac{1}{\lambda} \frac{\rho}{1-\rho}$$

$$= \frac{1}{\lambda}\left(\frac{\lambda/\mu}{1-\lambda/\mu}\right)$$

$$= \frac{1/\mu}{1-\lambda/\mu} = \underline{\frac{1}{\mu - \lambda}}$$

ii) W_q, the average time spent in the queue. [2]

$$W_q = W - \frac{1}{\mu}$$

$$= \frac{1}{\mu-\lambda} - \frac{1}{\mu}$$

$$= \frac{\mu}{\mu(\mu-\lambda)} - \frac{\mu-\lambda}{\mu(\mu-\lambda)}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{\mu-\lambda} = \underline{\frac{\rho}{\mu(1-\rho)}}$$

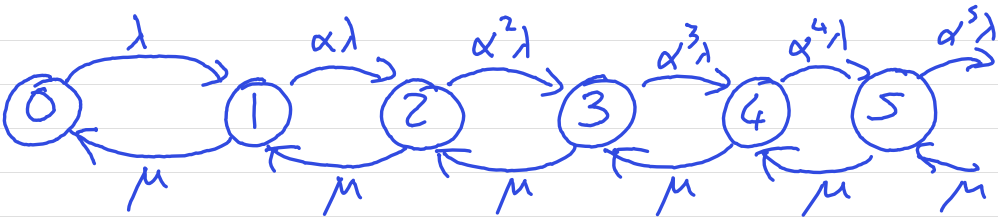
iii) L_q, the average number of customers waiting in the queue. [4]

From Little's law: $L_q = \lambda W_q$

$$L_q = \lambda\left(\frac{\rho}{\mu(1-\rho)}\right)$$

$$= \frac{\rho^2}{1-\rho}$$

Now consider an M/M/1 queue where customers baulk, that is choose not to join the queue. Customers baulk with probability $b(n) = \alpha^n$ when there are n customers already in the system upon arrival. The continuous-time Markov chain describing this system is given by:



d) Give an expression for the average holding time of state n, that is the average amount of time the system stays in state n before transitioning to another state. [3]

$$h(n) = \begin{cases} \lambda & \text{if } n = 0 \\ \alpha^n\lambda + \mu & \text{if } n > 0 \end{cases}$$

e) Show by induction that

$$P_n = \alpha^{T(n-1)} \rho^n P_0$$

For all n, where $T(n) = \sum_{K=0}^{n} K$, the sum of the first n integers. [8]

The steady-state probabilities are given by solving:

$$\lambda P_0 = \mu P_1 \quad \ldots\ldots\ (1)$$

$$(\alpha\lambda + \mu) P_1 = \lambda P_0 + \mu P_2 \quad \ldots\ldots\ (2)$$

$$(\alpha^2\lambda + \mu) P_2 = \alpha\lambda P_1 + \mu P_3 \quad \ldots\ldots\ (3)$$

$$\vdots$$

$$(\alpha^n\lambda + \mu) P_n = \alpha^{n-1}\lambda P_{n-1} + \mu P_{n+1} \quad \ldots\ldots\ (4)$$

$$\vdots$$

Now for the induction:

- Show it's true for $K=1$, that is $P_1 = \alpha^{T(1-1)} \rho^1 P_0 = \rho P_0$.

From (1):

$$\lambda P_0 = \mu P_1$$

$$\frac{\lambda}{\mu} P_0 = P_1$$

$$\underline{\rho P_0 = P_1}$$

- Assume $P_n = \alpha^{T(n-1)} \rho^n P_0$ is true for all n up to K.

- What about P_{K+1}?

From (4):

$$\alpha^{K-1} \lambda P_{K-1} + \mu P_{K+1} = (\alpha^K \lambda + \mu) P_K$$

$$\alpha^{K-1} \lambda \left[\alpha^{T(K-2)} \rho^{K-1} P_0\right] + \mu P_{K+1} = (\alpha^K \lambda + \mu)\left[\alpha^{T(K-1)} \rho^K P_0\right]$$

$$\lambda \alpha^{T(K-1)} \rho^{K-1} P_0 + \mu P_{K+1} = \lambda \alpha^{T(K)} \rho^K P_0 + \mu \alpha^{T(K-1)} \rho^K P_0$$

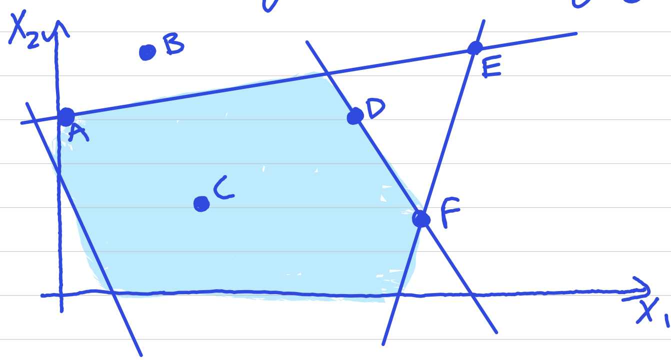
$$\mu \alpha^{T(K-1)} \rho^K P_0 + \mu P_{K+1} = \lambda \alpha^{T(K)} \rho^K P_0 + \mu \alpha^{T(K-1)} \rho^K P_0$$

$$\mu P_{K+1} = \lambda \alpha^{T(K)} \rho^K P_0$$

$$\underline{P_{K+1} = \alpha^{T(K)} \rho^{K+1} P_0}$$

Which completes the induction.

9) a) Consider the solution space (X_1, X_2) visualised below, with six constraints, including the non-negativity constraints. The feasible region is shaded in gray.



Six solutions are indicated. Categorise them by filling the table below:

	Basic	Non-Basic
Feasible	A, F	C, D
Infeasible	E	B

[6]

b) Re-write the following problem as a linear programming problem in standard form. You may need to introduce slack and/or dummy variables. [6]

Maximise $\min(X_1, X_2)$
subject to:

$$X_1 + X_2 \leq 9$$
$$5X_1 - 2X_2 \leq 12$$
$$X_1,\ X_2 \geq 0$$

[Note: You are not expected to solve the linear programming problem.]

Maximise y

subject to:

$$x_1 + x_2 + s_1 = 9$$
$$5x_1 - 2x_2 + s_2 = 12$$
$$y - x_1 + s_3 = 0$$
$$y - x_2 + s_4 = 0$$
$$x_1, x_2, y, s_1, s_2, s_3, s_4 \geq 0$$

c) Solve the following problem using the two-phase method:

Minimise $x_1 + 2x_2 + 3x_3$

subject to:

$$3x_1 + 2x_2 \geq 6$$
$$3x_1 + 2x_2 - 2x_3 \leq 1$$
$$-x_2 + x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

[12]

Constraints in standard form:

$$3x_1 + 2x_2 - s_1 + a_1 = 6$$
$$3x_1 + 2x_2 - 2x_3 + s_2 = 1$$
$$-x_2 + x_3 + s_3 = 8$$

the sum of artificial variables:

$$a_1 - 6 = -3x_1 - 2x_2 + s_1$$

Phase 1:

	x_1	x_2	x_3	s_1	s_2	s_3	a_1
6	3	2	0	-1	0	0	1
1	3	2	-2	0	1	0	0
8	0	-1	1	0	0	1	0
0	1	2	3	0	0	0	0
-6	-3 $\uparrow$	-2	0	1	0	0	0

$\tilde{r}_2 \to \frac{1}{3} r_2$ $\quad r_1 \to r_1 - r_2$ $\quad r_3 \to r_3$

$r_4 \to r_4 - \tilde{r}_2$ $\quad r_5 \to r_5 + r_2$

	x_1	x_2	x_3	s_1	s_2	s_3	a_1
5	0	0	2	-1	-1	0	1
1/3	1	2/3	-2/3	0	1/3	0	0
8	0	-1	1	0	0	1	0
-1/3	0	4/3	11/3	0	-1/3	0	0
-5	0	0	-2 $\uparrow$	1	1	0	0

$\hat{r}_1 \to \frac{1}{2} r_1$ $\quad r_2 \to r_2 + \frac{2}{3}\hat{r}_1$ $\quad r_3 \to r_3 - \hat{r}_1$

$r_4 \to r_4 - \frac{11}{3}\hat{r}_1$ $\quad r_5 \to r_5 + r_1$

	X_1	X_2	X_3	s_1	s_2	s_3	a_1
$5/2$	0	0	1	$-1/2$	$-1/2$	0	$1/2$
2	1	$2/3$	0	$-1/3$	0	0	$1/3$
$1/2$	0	-1	0	$1/2$	$1/2$	1	$-1/2$
$-19/2$	0	$4/3$	0	$11/6$	$3/2$	0	$-11/6$
0	0	0	0	0	0	0	1

$\therefore$ Phase 1 has ended, remove a_1 colum and artificial objective.

	X_1	X_2	X_3	s_1	s_2	s_3
$5/2$	0	0	1	$-1/2$	$-1/2$	0
2	1	$2/3$	0	$-1/3$	0	0
$1/2$	0	-1	0	$1/2$	$1/2$	1
$-19/2$	0	$4/3$	0	$11/6$	$3/2$	0

$\therefore$ Phase 2 has ended, read off solution as

$X_1 = 2, \; X_2 = 0, \; X_3 = 5/2$

d) Give an example of software used to solve linear programming problems. [1]

Gurobi / CPLEX / PuLP

10) There are three factories that produce caster sugar (A, B, and C), each can produce 9 tonnes of sugar a month. There are three bakeries that order the sugar (X, Y, and Z), demanding 7·3 tonnes, 10·5 tonnes, and 12·1 tonnes of sugar per month, respectively. The route costs are given in the table below:

	A	B	C
X	2	7	2
Y	4	11	6
Z	7	3	1

We wish to find an allocation that minimises the total transportation cost.

a) Letting X_{ij} be the amount of goods to transport from factory i to bakery j, formulate the problem as a linear programming problem, writing out the objective function and all constraints. [6]

[Note: you are not expected to solve the linear programming problem]

Minimise $2X_{AX} + 7X_{BX} + 2X_{CX} + 4X_{AY} + 11X_{BY}$
$+ 6X_{CY} + 7X_{AZ} + 3X_{BZ} + X_{CZ}$

Subject to:

$$X_{AX} + X_{AB} + X_{AZ} \leq 9$$
$$X_{BX} + X_{BY} + X_{BZ} \leq 9$$
$$X_{CX} + X_{CY} + X_{CZ} \leq 9$$
$$X_{AX} + X_{BX} + X_{CX} \geq 7{\cdot}3$$
$$X_{AY} + X_{BY} + X_{CY} \geq 10{\cdot}5$$
$$X_{AZ} + X_{BZ} + X_{CZ} \geq 12{\cdot}1$$
$$X_{AX}, X_{AY}, X_{AZ}, X_{BX}, X_{BY}, X_{BZ}, X_{CX}, X_{CY}, X_{CZ} \geq 0.$$

b) Use the minimum cost rule to find a feasible basic solution. (Note that the problem may be infeasible, so a dummy factory may need to be used). [4]

	A	B	C	Dummy	
X	7.3 (2)	(7)	(2)	(0)	7.3
Y	1.7 (4)	5.9 (11)	(6)	2.9 (0)	10.5
Z	(7)	3.1 (3)	9 (1)	(0)	12.1
	9	9	9	2.9	

c) Use the stepping-stone algorithm to find an optimal solution. [5]

Non-Basic Variable	Cost
X_B	$7-11+4-2 = -2$
X_C	$2-1+3-11+4-2 = -5$
X_D	$0-6+4-2 = -2$
YC	$6-1+3-11 = -3$
ZA	$7-4+11-3 = 11$
ZD	$0-3+11-0 = 8$

$\therefore$ Increase X_C by 5.9

	A	B	C	Dummy	
X	1.4 (2)	(7)	5.9 (2)	(0)	7.3
Y	7.6 (4)	(11)	(6)	2.9 (0)	10.5
Z	(7)	9 (3)	3.1 (1)	(0)	12.1
	9	9	9	2.9	

Non-Basic Variable	Cost
XB	$7-3+1-2=3$
XD	$0-0+4-2=2$
BY	$11-3+1-2+2-4=5$
BC	$6-2+2-4=2$
AZ	$7-1+2-2=6$
ZD	$0-0+4-2+2-1=3$

∴ An optimal solution has been found.

d) What is the cost of the optimal allocation? [1]

$$\text{Cost} = (1.4\times2)+(5.9\times2)+(7.6\times4)+(2.9\times0)+(9\times3)+(3.1\times1)$$

$$\text{Cost} = 75.1$$

e) Bakery Y is considering charging a fine for each tonne of demand not satisfied. What is the maximum fine they could charge before the solution found in part (c) is no longer optimal? [4]

Let the cost of route YD be F, a fine.

Then:

Non-Basic Variable	Cost
XB	$7-3+1-2=3$
XD	$0-F+4-2=2-F$
BY	$11-3+1-2+2-4=5$
BC	$6-2+2-4=2$
AZ	$7-1+2-2=6$
ZD	$0-F+4-2+2-1=3-F$

∴ To remain optimal we require

$$2 - F > 0 \quad \Rightarrow \quad 2 > F$$
and
$$3 - F > 0 \qquad \text{and} \quad 3 > F$$

$$\therefore \quad F < 2$$

F) Bakery X decides to boycott factory A over reports of child labor, therefore route XA becomes unavailable. Find a new optimal route. [5]

Set the cost of route XA to an arbitrary big number, M.

	A	B	C	Dummy	
X	1.4 M	7	5.9 2	0	7.3
Y	7.6 4	11	6	2.9 0	10.5
Z	7	9 3	3.1 1	0	12.1
	9	9	9	2.9	

So:

Non-Basic Variable	Cost
XB	$7-3+1-2 = 3$
XD	$0-0+4-M = 4-M$
BY	$11-3+1-2+M-4 = 3+M$
CY	$6-2+M-4 = M$
AZ	$7-1+2-M = 8-M$
ZD	$0-0+4-M+2-1 = 5-M$

∴ The biggest saving is XD, so increase XD by 1.4.

	A	B	C	Dummy	
X	$_M$	$_7$	5.9 $_2$	1.4 $_0$	7.3
Y	9 $_4$	$_{11}$	$_6$	1.5 $_0$	10.5
Z	$_7$	9 $_3$	3.1 $_1$	$_0$	12.1
	9	9	9	2.9	

Non-Basic Variable	Cost
XA	Increases XA
XB	$7-2+1-3=3$
YB	$11-3+1-2+0-0=7$
YC	$6-2+0-0=4$
AZ	$7-4+0-0+2-1=4$
DZ	$0-0+2-1=1$

$\therefore$ A new optimal solution has been found that does not use route XA.