Solutions to Problem Sheet 1

1. Apples falling off a tree randomly in time can be described as a Poisson process with rate $\lambda = 2$ apples per day.

   (a) What is the probability that less than 3 apples fall in a day?

   (b) What is the probability that less than 3 apples fall in a week?

   (c) What is the average time between two consecutive apples falling?

   (d) What is the probability that I wait longer than 12 hours for an apple to fall?

Solution 1 *We have three random variables of interest, $X_d \sim$ Poisson(2) is the number of apples falling in a day, $X_w \sim$ Poisson(14) is the number of apples falling in a week, and $T \sim$ Expon(2) is the time between two consecutive apples falling. Then:*

(a) The probability that less than 3 apples fall in a day:

$$\begin{aligned}\mathbb{P}(X_d < 3) &= \mathbb{P}(X_d = 0) + \mathbb{P}(X_d = 2) + \mathbb{P}(X_d = 2) \\ &= \left(\frac{2^0 e^{-2}}{0!}\right) + \left(\frac{2^1 e^{-2}}{1!}\right) + \left(\frac{2^2 e^{-2}}{2!}\right) \\ &= e^{-2} + 2e^{-2} + 2e^{-2} \\ &= 0.67667\end{aligned}$$

(b) The probability that less than 3 apples fall in a week:

$$\begin{aligned}\mathbb{P}(X_w < 3) &= \mathbb{P}(X_w = 0) + \mathbb{P}(X_w = 2) + \mathbb{P}(X_w = 2) \\ &= \left(\frac{14^0 e^{-14}}{0!}\right) + \left(\frac{14^1 e^{-14}}{1!}\right) + \left(\frac{14^2 e^{-14}}{2!}\right) \\ &= e^{-14} + 14e^{-14} + 196e^{-14} \\ &= 0.00018\end{aligned}$$

(c) The average time between two consecutive apples falling:

$$\mathbb{E}[T] = \frac{1}{2}$$

(d) The probability that I wait longer than 12 hours for an apple to fall:

$$\begin{aligned}\mathbb{P}(T > 1/2) &= 1 - \mathbb{P}(T < 1/2) \\ &= 1 - \left(1 - e^{-2\times 1/2}\right) \\ &= e^{-1} = 0.36788\end{aligned}$$

2. During election season political placards are places randomly along the length of the A470, which can be described as a Poisson process in space with rate $\lambda = 3/8$ per mile. 25% of the placards are from the Red party, 40% are from the Yellow party, and 35% are from the Blue party.

   (a) I drive a strech of 55 miles, how many Yellow placards should I expect to see?

   (b) What is the probability of not seeing any Blue placards for 20 miles?

   (c) How long would I have to drive before the probability of having seen a Red placard is greater than 90%?

Solution 2 *Let $X \sim$ Poisson$(3/8)$ be the number of placards seen in a mile, and let $T \sim$ Expon$(3/8)$ be the distance between two consecutive placards. Due to the thinning of Poisson processes, we also have:*

- $X_R \sim$ *Poisson$(3/32)$ is the number of placards of the red party per mile;*
- $T_R \sim$ *Expon$(3/32)$ is the distance between two consecutive of placards from the red party;*
- $X_Y \sim$ *Poisson$(3/20)$ is the number of placards of the yellow party per mile;*
- $T_Y \sim$ *Expon$(3/20)$ is the distance between two consecutive of placards from the yellow party;*
- $X_B \sim$ *Poisson$(21/160)$ is the number of placards of the blue party per mile;*
- $T_B \sim$ *Expon$(21/160)$ is the distance between two consecutive of placards from the blue party;*

Therefore:

(a) The expected number of yellow placards in 55 miles:

$$\mathbb{E}[X_Y] = 55 \times 3/20 = 33/4$$

(b) The probability of not seeing any Blue placards for 20 miles:

$$\begin{aligned}\mathbb{P}(T_B > 20) &= 1 - \mathbb{P}(T_B < 20) \\ &= 1 - \left(1 - e^{20 \times 21/160}\right) \\ &= 0.07244\end{aligned}$$

(c) The distance t to drive so that $0.9 = \mathbb{P}(T_R < t)$:

$$\begin{aligned}0.9 &= \mathbb{P}(T_R < t) \\ 0.9 &= 1 - e^{3/32t} \\ e^{3/32t} &= 1 - 0.9 \\ \frac{3}{32}t &= \ln(0.1) \\ t &= \frac{32}{3}\ln(0.1) \\ t &= 24.5609\end{aligned}$$