Solutions to Problem Sheet 3

1. Describe in words the following queueing systems:
   (a) $M/M/5$
   (b) $D/M/1/5/$SIRO
   (c) $G/G/\infty$
   (d) $M^3/D/\infty/\infty/$PS
   (e) $M/E_2/1/\infty/$FIFO

Solution 1 *We have:*

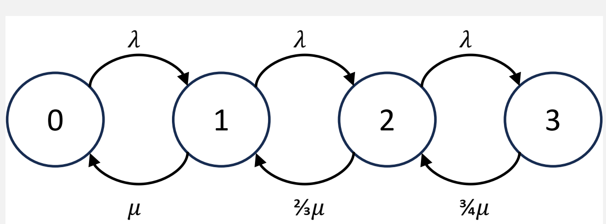
(a) $M/M/5$: Markovian arrivals, Markovian services, 5 servers, infinite capacity, first-in-first-out.
(b) $D/M/1/5/$SIRO: Deterministic arrivals, Markovian services, single server, system capacity of 5, service-in-random-order.
(c) $G/G/\infty$: General arrival distribution, general service distribution, infinite servers, infinite capacity, first-in-first-out.
(d) $M^3/D/\infty/\infty/$PS: Markovian arrivals with batch size of 3, Deterministic service times, infinite servers, infinite capacity, processor-sharing.
(e) $M/E_2/1/\infty/$FIFO: Markovian arrivals, services with Erlang distirbution parameter 2, single server, infinite capacity, first-in-first-out.

2. At a chicken shop the deep-fat-fryer can be described as a Markovian queue. Chicken pieces are put in the fryer at a rate of λ per time unit. There is room for only 3 pieces of chicken in the fryer, and no new chicken orders are taken when the fryer is full. When a piece of chicken if in the fryer alone it completes frying at a rate of μ. When there are two pieces of chicken in the fryer they cook at a rate of $\mu/3$, and when there are three pieces of chicken in the fryer they cook at a rate of $\mu/4$.

   Find the steady-state probabilities in terms of λ and μ.

   Find the expected number of pieces of chicken in the fryer when $\lambda = 5$ and $\mu = 9$.

Solution 2 *We have:*



Solution 2 (continuing from p. 1) *Solving for steady state we have:*

$$\begin{aligned}
\lambda\pi_0 &= \mu\pi_1 \\
(\lambda+\mu)\,\pi_1 &= \lambda\pi_0 + \frac{2}{3}\mu\pi_2 \\
\left(\lambda + \frac{2}{3}\mu\right)\pi_2 &= \lambda\pi_1 + \frac{3}{4}\mu\pi_3 \\
\frac{3}{4}\mu\pi_3 &= \lambda\pi_2 \\
\pi_0+\pi_1+\pi_2+\pi_3 &= 1
\end{aligned}$$

Solving π_1 in terms of π_0:

$$\pi_1 = \frac{\lambda}{\mu}\pi_0$$

Solving π_2 in terms of π_0:

$$\begin{aligned}
(\lambda+\mu)\pi_1 &= \lambda\pi_0 + \frac{2}{3}\mu\pi_2 \\
(\lambda+\mu)\frac{\lambda}{\mu}\pi_0 &= \lambda\pi_0 + \frac{2}{3}\mu\pi_2 \\
\frac{\lambda^2}{\mu}\pi_0 &= \frac{2}{3}\mu\pi_2 \\
\pi_2 &= \frac{3\lambda^2}{2\mu^2}\pi_0
\end{aligned}$$

Solving π_3 in terms of π_0:

$$\begin{aligned}
\pi_3 &= \frac{4\lambda}{3\mu}\pi_2 = \frac{4\lambda}{3\mu}\left(\frac{3\lambda^2}{2\mu^2}\pi_0\right) \\
\pi_3 &= \frac{2\lambda^3}{\mu^3}\pi_0
\end{aligned}$$

Finally we have, for π_0:

$$\begin{aligned}
\pi_0+\pi_1+\pi_2+\pi_3 &= 1 \\
\pi_0\left(1 + \frac{\lambda}{\mu} + \frac{3\lambda^2}{2\mu^2} + \frac{2\lambda^3}{\mu^3}\right) &= 1 \\
\pi_0 &= \frac{1}{\left(1 + \frac{\lambda}{\mu} + \frac{3\lambda^2}{2\mu^2} + \frac{2\lambda^3}{\mu^3}\right)}
\end{aligned}$$

Solution 2 (continuing from p. 2) *When $\lambda = 5$ and $\mu = 9$ we have:*

$$\pi_0 = \frac{1458}{3443} \qquad \pi_1 = \frac{810}{3443} \qquad \pi_2 = \frac{675}{3443} \qquad \pi_3 = \frac{500}{3443}$$

And so the expected number of pieces of chicken in the fryer is:

$$\begin{aligned} L &= \left(0 \times \frac{1458}{3443}\right) + \left(1 \times \frac{810}{3443}\right) + \left(2 \times \frac{675}{3443}\right) + \left(3 \times \frac{500}{3443}\right) \\ &= \frac{3660}{3443} = 1.063 \end{aligned}$$

3. Consider an $M/M/1$ queue with arrival rate $\lambda = 10$ and $\mu = 15$. Find ρ, P_0, L, W, W_q, and L_q.

Solution 3 *We can use the formulae:*

$$\begin{aligned} \rho &= \frac{\lambda}{\mu} = \frac{10}{15} = \frac{2}{3} \\ P_0 &= 1 - \rho = 1 - \frac{2}{3} = \frac{1}{3} \\ L &= \frac{\rho}{1-\rho} = \frac{2/3}{1 - 2/3} = 2 \\ W &= \frac{1}{\lambda} L = \frac{1}{10} \times 2 = \frac{1}{5} \\ W_q &= W - \frac{1}{\mu} = \frac{1}{5} - \frac{1}{15} = \frac{2}{15} \\ L_q &= \lambda W_q = 10 \times \frac{2}{15} = \frac{4}{3} \end{aligned}$$

4. Consider an $M/M/\infty$ queue with arrival rate $\lambda = 6$ and $\mu = 8$. Find P_0, L, W, W_q, and L_q.

Solution 4 *We can use the formulae and some heuristic thinking:*

$$\begin{aligned} P_0 &= e^{-\theta} = e^{-6/8} = 0.47237 \\ W &= \frac{1}{\mu} = \frac{1}{8} \\ W_q &= L_q = 0 \quad \textit{as no-one queues} \\ L &= \lambda W = 6 \times \frac{1}{8} = \frac{3}{4} \end{aligned}$$

5. For an $M/M/\infty$ queue with arrival rate λ and service rate μ, derive the fact that $L = {}^{\lambda}/_{\mu}$ without using Little's laws.

Solution 5 *We know that if $\theta = {}^{\lambda}/_{\mu}$, then:*

$$P_k = \frac{\theta^k}{k!}P_0 \qquad\qquad P_0 = e^{-\theta}$$

Now consider L:

$$\begin{aligned}
L &= \sum_{k=0}^{\infty} kP_k \\
&= \sum_{k=1}^{\infty} kP_k \\
&= \sum_{k=1}^{\infty} k\frac{\theta^k}{n!}e^{-\theta} \\
&= e^{-\theta}\sum_{k=1}^{\infty} \frac{\theta^k}{(k-1)!} \\
&= \theta e^{-\theta}\sum_{k=1}^{\infty} \frac{\theta^{k-1}}{(k-1)!} \\
&= \theta e^{-\theta}\sum_{n=0}^{\infty} \frac{\theta^n}{n!} \\
&= \theta e^{-\theta}e^{\theta} \\
&= \theta = {}^{\lambda}/_{\mu}
\end{aligned}$$

by recognising the series expansion of the exponential function.

6. A blood diagnostic centre can be described as an $M/M/1$ queue. Blood samples arrive at a rate of λ per time unit. Once in the centre, whether being processed of waiting, the samples need to be kept cold, at a cost of C_h per time unit. An automated diagnostic machine can process the blood samples one at a time, at a rate of μ per time unit, which can be controlled. It costs μC_s per time unit to run the machine at a rate μ. What should the machine's service rate be set to in order to minimise the overall cost?

Solution 6 *The overall cost per time unit would be:*

$$
\begin{aligned}
C &= C_h L + \mu C_s \\
&= C_h \frac{\rho}{1-\rho} + \mu C_s \\
&= \frac{C_h \lambda}{\mu - \lambda} + \mu C_s \\
&= C_h \lambda (\mu - \lambda)^{-1} + \mu C_s
\end{aligned}
$$

To minimise this, we look for the value of μ where $\frac{dC}{d\mu} = 0$:

$$
\begin{aligned}
\frac{dC}{d\mu} &= -\lambda C_h (\mu - \lambda)^{-2} + C_s \\
0 &= -\lambda C_h (\mu - \lambda)^{-2} + C_s \\
\frac{\lambda C_h}{(\mu - \lambda)^2} &= C_s \\
\frac{\lambda C_h}{C_s} &= (\mu - \lambda)^2 \\
\sqrt{\frac{\lambda C_h}{C_s}} &= \mu - \lambda \\
\mu &= \sqrt{\frac{\lambda C_h}{C_s}} + \lambda
\end{aligned}
$$

where we took the positive root as we know that $\mu > \lambda$ due to stability.
To check this is a minimum and not a maximum, use the second derivative test:

$$
\begin{aligned}
\frac{d^2C}{d\mu^2} &= 2\lambda C_h \left(\mu - \lambda\right)^{-3} \\
&= 2\lambda C_h \left(\sqrt{\frac{\lambda C_h}{C_s}}\right)^{-3} \\
&\geq 0
\end{aligned}
$$

so $\mu = \sqrt{\frac{\lambda C_h}{C_s}} + \lambda$ is a minimum.