Solutions to Problem Sheet 5

1. A toothpaste company can make two kinds of toothpaste:

   - 'Sure Smiles': a budget toothpaste that makes a profit of £1000 a tonne, and
   - 'Wicked Whites': a premium toothpaste that makes of profit of £8000 a tonne.

   Two of the ingredients need to be imported and so their daily use is limited: only 12 kilograms of Calcium Carbonate can be used each day, and only 24 kilograms of Sodium Fluoride can be used each day.

   - Each tonne of 'Sure Smiles' requires 3 kilograms of Sodium Fluoride and 1 kilogram of Calcium Carbonate.
   - Each tonne of 'Wicked Whites' requires 1 kilogram of Sodium Fluoride and 2 kilograms of Calcium Carbonate.

   Additionally, to ensure that there is enough budget toothpaste available to the population, the government has legislated that the company cannot produce more than 2 tonnes more of premium toothpaste than the budget toothpaste each day.

   (a) Using the graphical method, how many tonnes of each toothpaste should the company produce each day to maximise their daily profit?

   (b) If the government now legislates that the company can only make £1600 per tonne of 'Wicked Whites', how many tonnes of each toothpaste should the company produce each day to maximise their daily profit now?

Solution 1 *Let S be the number of tonnes of 'Sure Smiles' and W be the number of tonnes of 'Wicked Whites'. Then:*

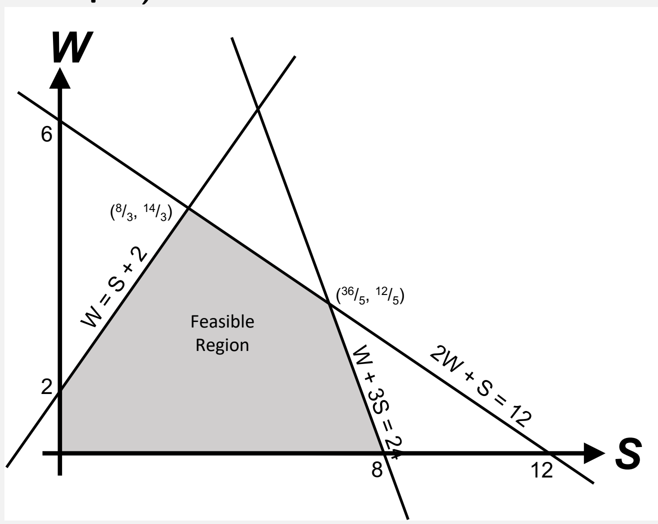
Maximise:

$$800W + 100S$$

subject to

$$\begin{aligned} W + 3S &\leq 24 \\ 2W + S &\leq 12 \\ B &\leq A + 2 \\ W, S &\geq 0 \end{aligned}$$

Solution 1 (continuing from p. 1) *This is drawn as:*



(a) *When the objective is £8000 W + £1000 S:*

Point	*Objective* $= 8000W + 1000S$
$(0, 0)$	0
$(0, 2)$	$16,000$
$(8, 0)$	$8,000$
$(^8/_3, ^{14}/_3)$	$40,000$
$(^{36}/_5, ^{12}/_5)$	$26,400$

So $W = {}^{14}/_3$ *and* $S = {}^8/_3$.

(b) *When the objective is £1600 W + £1000 S:*

Point	*Objective* $= 1600W + 1000S$
$(0, 0)$	0
$(0, 2)$	$3,200$
$(8, 0)$	$8,000$
$(^8/_3, ^{14}/_3)$	$10,133.33$
$(^{36}/_5, ^{12}/_5)$	$11,040$

So $W = {}^{12}/_5$ *and* $S = {}^{36}/_5$.

2. Use the Simplex method to solve the following Linear programming problem:

Maximise:

$$3x_1 + 5x_2$$

subject to

$$-5x_1 + 17x_2 \leq 425$$
$$5x_1 + 4x_2 \leq 205$$
$$x_1, x_2 \geq 0$$

Solution 2 *Setting up the initial Simplex tableau:*

	x_1	x_2	s_1	s_2
425	-5	(17)	1	0
205	5	4	0	1
0	-3	-5	0	0

Choosing 17 as the pivot, we perform $\bar{r}_1 = \frac{1}{17}r_1$, $r_2 = r_2 - 4\bar{r}_1$, *and* $r_3 = r_3 + 5\bar{r}_1$*:*

	x_1	x_2	s_1	s_2
25	$-5/17$	1	$1/17$	0
105	($105/17$)	0	$-4/17$	1
125	$-76/17$	0	$5/17$	0

Choosing $105/17$ *as the pivot, we perform* $\bar{r}_2 = \frac{17}{105}r_2$, $r_1 = r_1 + \frac{5}{17}\bar{r}_2$, *and* $r_3 = r_3 + \frac{76}{17}\bar{r}_1$*:*

	x_1	x_2	s_1	s_2
30	0	1	$1/21$	$1/21$
17	1	0	$-4/105$	$17/105$
201	0	0	$13/105$	$76/105$

Giving a solution of $x_1 = 17$, $x_2 = 30$, *and a maximum objective value of* $3x_1 + 5x_2 = 201$.

3. Consider the following linear programming problem:

Maximise:

$$3x_1 + x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 - x_2 + 4x_3 &\leq 17 \\ 2x_1 + x_3 &\leq 6 \\ 2x_2 + 3x_3 &\leq 14 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Use the Simplex method to find one optimal solution.

(b) Pivot one more time to find all optimal solutions.
Give your answer in the form $\{(1-t)\underline{\mathbf{a}} + t\underline{\mathbf{b}}$ for all $t \in [0,1]\}$.

(c) If we fix $x_3 = 1$, find the values that x_1 and x_2 must take for the solution to remain optimal.

Solution 3 *We have:*

(a) Setting up the initial Simplex tableau:

	x_1	x_2	x_3	s_1	s_2	s_3
17	1	-1	4	1	0	0
6	(2)	0	1	0	1	0
14	0	2	3	0	0	1
0	-3	-1	-3	0	0	0

Choosing 2 as the pivot, we perform $\bar{r}_2 = \frac{1}{2}r_2$, $r_1 = r_1 - \bar{r}_2$, $r_3 = r_3$, *and* $r_4 = r_4 + 3\bar{r}_2$*:*

	x_1	x_2	x_3	s_1	s_2	s_3
14	0	-1	($7/2$)	1	$-1/2$	0
3	1	0	$1/2$	0	$1/2$	0
14	0	2	3	0	0	1
9	0	-1	$-3/2$	0	$3/2$	0

Choosing $7/2$ *as the pivot, we perform* $\bar{r}_1 = \frac{2}{7}r_1$, $r_2 = r_2 - \frac{1}{2}\bar{r}_1$, $r_3 = r_3 - 3\bar{r}_1$, *and* $r_4 = r_4 + \frac{3}{2}\bar{r}_1$*:*

	x_1	x_2	x_3	s_1	s_2	s_3
4	0	$-2/7$	1	$2/7$	$-1/7$	0
1	1	$1/7$	0	$-1/7$	$4/7$	0
2	0	($20/7$)	0	$-6/7$	$-3/7$	1
15	0	$-10/7$	0	$3/7$	$9/7$	0

Choosing $20/7$ *as the pivot, we perform* $\bar{r}_3 = \frac{7}{20}r_3$, $r_1 = r_1 - \frac{1}{10}r_3$, $r_2 = r_2 - \frac{1}{20}r_3$, *and* $r_4 = r_4 + \frac{1}{2}r_3$*:*

	x_1	x_2	x_3	s_1	s_2	s_3
$21/5$	0	0	1	$1/5$	$-13/70$	$1/10$
$9/10$	1	0	0	$-1/10$	$11/20$	$-1/20$
$7/10$	0	1	0	$-3/10$	$-3/20$	$-7/20$
16	0	0	0	0	$15/14$	$1/2$

And so an optimal solution is $x_1 = 9/10$, $x_2 = 7/10$, *and* $x_3 = 21/5$, *giving an objective function value of 16.*

(b) As there is a non-basic variable with a zero in the objective row (s_1)*, we pivot one more time on* $1/5$*. Choosing* $\bar{r}_1 = 5r_1$, $r_2 = r_2 + \frac{1}{2}r_1$, $r_3 = r_3 + \frac{3}{2}r_1$, *and* $r_4 = r_4$*:*

	x_1	x_2	x_3	s_1	s_2	s_3
21	0	0	5	1	$-13/14$	$1/2$
3	1	0	$1/2$	0	$5/28$	$3/20$
7	0	1	$3/2$	0	$-13/14$	$1/2$
16	0	0	0	0	$15/14$	$1/2$

And so another optimal solution is $x_1 = 3$, $x_2 = 7$, *and* $x_3 = 21$.

We then write all optimal solutions in the form:

$$\left\{(1-t)\left(\frac{9}{10}, \frac{7}{10}, \frac{21}{5}\right) + t\,(3,7,0) \quad \textit{for all } \; t \in [0,1]\right\}$$

Solution 3 (continuing from p. 4) *(c) Fixing $x_3 = 1$ corresponds to setting $t = {}^{16}/_{21}$, this gives:*

$$\begin{aligned} x_1, x_2, x_3 &= \left(1 - \frac{16}{21}\right)\left(\frac{9}{10}, \frac{7}{10}, \frac{21}{5}\right) + \frac{16}{21}(3, 7, 0) \\ x_1, x_2, x_3 &= \frac{5}{21}\left(\frac{9}{10}, \frac{7}{10}, \frac{21}{5}\right) + \frac{16}{21}(3, 7, 0) \\ &= \left(\frac{45}{210}, \frac{35}{210}, 1\right) + \left(\frac{48}{21}, \frac{112}{21}, 0\right) \\ &= \left(\frac{525}{210}, \frac{1155}{210}, 1\right) \\ &= \left(\frac{5}{2}, \frac{11}{2}, 1\right) \end{aligned}$$

And so the optimal solution would now be $x_1 = {}^5/_2$, $x_2 = {}^{11}/_2$, and $x_3 = 1$.

4. Solve the following linear programming problem using the two-phase method:

Maximise:

$$2x_1 + 3x_2 + 4x_3$$

subject to

$$\begin{aligned} 3x_1 + 2x_2 + x_3 &\le 10 \\ 2x_1 + 3x_2 + 3x_3 &\le 15 \\ x_1 + x_2 - x_3 &\ge 4 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

Solution 4 *Re-writing the constraints using slack and artificial variables, we get:*

$$\begin{aligned} 3x_1 + 2x_2 + x_3 + s_1 &= 10 \\ 2x_1 + 3x_2 + 3x_3 + s_2 &= 15 \\ x_1 + x_2 - x_3 - s_3 + a_1 &= 4 \end{aligned}$$

and so the first phase is to minimise $a_1 - 4 = -x_1 - x_2 + x_3 + s_3$.

Solution 4 (continuing from p. 5) *Writing the tableau gives:*

	x_1	x_2	x_3	s_1	s_2	s_3	a_1
10	(3)	2	1	1	0	0	0
15	2	3	3	0	1	0	0
4	1	1	-1	0	0	-1	1
0	-2	-3	-4	0	0	0	0
-4	-1	-1	1	0	0	1	0

Choosing 3 *as the pivot, we perform* $\bar{r}_1 = \frac{1}{3}r_1$, $r_2 = r_2 - 2\bar{r}_1$, $r_3 = r_3 - \bar{r}_1$, $r_4 = r_4 + 2\bar{r}_1$, *and* $r_5 = r_5 + \bar{r}_1$*:*

	x_1	x_2	x_3	s_1	s_2	s_3	a_1
$^{10}/_3$	1	$^2/_3$	$^1/_3$	$^1/_3$	0	0	0
$^{25}/_3$	0	$^5/_3$	$^7/_3$	$-^2/_3$	1	0	0
$^2/_3$	0	($^1/_3$)	$-^4/_3$	$-^1/_3$	0	-1	1
$^{20}/_3$	0	$-^5/_3$	$-^{10}/_3$	$^2/_3$	0	0	0
$-^2/_3$	0	$-^1/_3$	$^4/_3$	$^1/_3$	0	1	0

Choosing $^1/_3$ *as the pivot, we perform* $\bar{r}_3 = 3r_3$, $r_1 = r_1 - 2r_3$, $r_2 = r_2 - 5r_3$, $r_4 = r_4 + 5r_3$, *and* $r_5 = r_5 + r_3$*:*

	x_1	x_2	x_3	s_1	s_2	s_3	a_1
2	1	0	3	1	0	2	-2
5	0	0	9	1	1	5	-5
2	0	1	-4	-1	0	-3	3
10	0	0	-10	-1	0	-5	5
0	0	0	0	0	0	0	1

This ends the first phase. Deleting the appropriate columns and rows gives:

	x_1	x_2	x_3	s_1	s_2	s_3
2	1	0	3	1	0	2
5	0	0	(9)	1	1	5
2	0	1	-4	-1	0	-3
10	0	0	-10	-1	0	-5

Choosing 9 *as the pivot, we perform* $\bar{r}_2 = \frac{1}{9}r_2$, $r_1 = r_1 - 3\bar{r}_2$, $r_3 = r_3 + 4\bar{r}_2$, *and* $r_4 = r_4 + 10\bar{r}_2$*:*

	x_1	x_2	x_3	s_1	s_2	s_3
$^1/_3$	1	0	0	$^2/_3$	$-^1/_3$	$^1/_3$
$^5/_9$	0	0	1	$^1/_9$	$^1/_9$	$^5/_9$
$^{38}/_9$	0	1	0	$-^5/_9$	$^4/_9$	$-^7/_9$
$^{140}/_9$	0	0	0	$^1/_9$	$^{10}/_9$	$^5/_9$

And so the optimal solution is $x_1 = {}^1/_3$, $x_2 = {}^{38}/_9$, *and* $x_3 = {}^5/_9$, *giving a maximum value of the objective function of* $^{140}/_9$.

5. Cardiff University needs to create its exam timetable. It has a set M of exams (indexed by m) to schedule. For each pair of exams i, j, it has an indicator C_{ij} that is set to 1 if the modules cannot be scheduled at the same time (due to sharing students), and 0 if they can be scheduled at the same time. Let T be the set of time slots available, indexed by t. Formulate an linear programming problem that finds a feasible schedule using the least time slots.

   You are not asked to solve the linear programming problem!

Solution 5 *Define X_{mt} as a binary variable indicating if module $m \in M$ is scheduled at time $t \in T$. Define Y_t as the binary variable indicating if there is an exam on day $t \in T$. Then a possible formulation would be:*

Minimise:

$$\sum_{t \in T} Y_t$$

subject to

$$\sum_{t \in T} X_{mt} = 1 \; \forall \, m \in M$$

$$|M| Y_t \geq \sum_{m \in M} X_{mt} \; \forall \, t \in T$$

$$C_{ij} \left(X_{it} + X_{jt} \right) \leq 1 \; \forall \, t \in T \; \forall \, i, j \in M$$

$$X_{mt}, Y_t \text{ is binary} \; \forall \, t \in T \; \forall \, m \in M$$