Simulation Modelling with Python

Example A: Falling Fridge Magnets

You have a fridge with fridge magnets on that spell the words **HELLO WORLD**. An earthquake strikes and two letters fall off the fridge at random. What is the probability that one of those letters is a vowel?

Task 1: Weight of a Shopping Bag

When grocery shopping, I buy apples with probability 0.4 and melons with probability 0.2. I roll a die to see how many of each fruit I buy. The weight of an apple is Uniformly distributed between 70 and 100 grams. The weight of a melon is Normally distributed with mean 1200g and standard deviation 250g.

1. What is the expected weight of the shopping bag?
2. Explore the effect of the number of repetitions on the expected weight.
3. What is the probability of the bag exceeding 6kg?
4. What does its weight distribution of the bag look like?

Example B: Evaluating an Integral

Estimate the following integral using the Monte Carlo method:

$$I = \int_{3}^{11} \left| \sin\left(\frac{x}{10}\right) \cos 2x \right| dx$$

Example C: An M/M/1 Queue

There is a very popular arcade machine in the University staff room, so popular that staff will happily wait in a line to use the machine. When a member of staff wants to play on the machine, if there are no other staff present, they begin using the machine immediately. If there is another member of staff already playing on the machine, they will wait and form a queue to play. When someone finishes playing, the person at the head of the queue can then begin playing.

Staff who want to play on the machine arrive randomly at a rate of 11 per hour. The times between each consecutive arrival is Exponentially distributed. The amount of time staff spend on the machine is random, and is Exponentially distributed, lasting on average 5 minutes.

1. How long on average to staff spend waiting to play on the arcade machine?
2. What will be the effect of investing in a second machine?

Task 2: An M/G/c Queue

Assume you are a bank manager and would like to know how long customers wait in your bank. Customers arrive randomly, roughly 12 per hour with Exponentially distributed inter-arrival times. Service times are random, also Uniformly distributed, lasting between 5 and 17 minutes. The bank is open 24 hours a day, 7 days a week, and has three servers who are always on duty. If all servers are busy, newly arriving customers form a queue and wait for service.

1. On average how long do customers wait?
2. How many customers wait over 10 minutes?
3. How many customers wait over 1 minute?

Example D: A Network of Queues

At an Emergency Department patients arrive randomly, with Exponentially distributed inter-arrival times, at a rate of one every minute. They first queue up for reception to register to be seen. They then enter another waiting room to be seen by a clinician. In rare cases (5% of the time), a clinician will send the patient back to reception to re-register, as they would need further consultation with another clinician. It takes exactly 2.2 minutes to register at reception, and consultation lengths are Uniformly distributed between 5 and 28 minutes. There are 3 receptionists and 18 clinicians.

1. On average how long to patients wait for consultation?
2. How does this change if an extra clinician is hired?

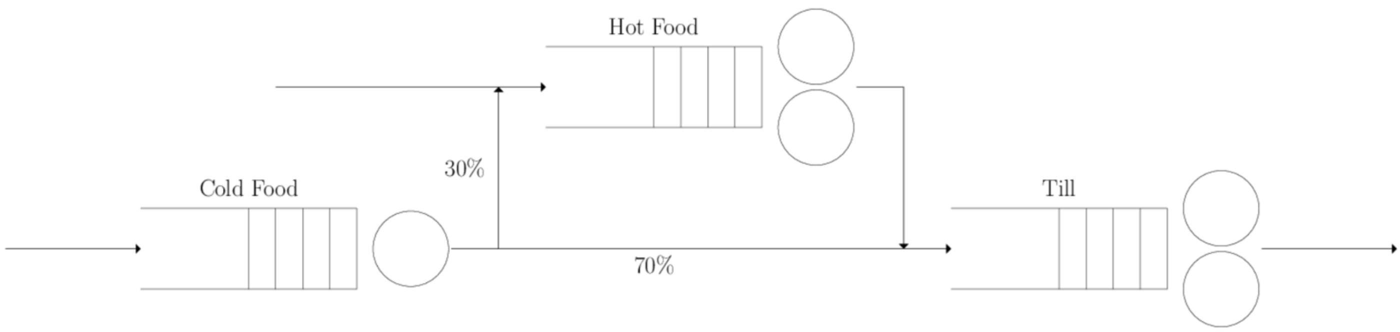
Task 3: A Café

A café that sells both hot and cold food. Customers arrive and can take a few routes:

- Customers only wanting cold food must queue at the cold food counter, and then take their food to the till to pay.
- Customers only wanting hot food must queue at the hot food counter, and then take their food to the till to pay.
- Customers wanting both hot and cold food must first queue for cold food, then hot food, and then take both to the till and pay.

In this system there are three nodes: Cold food counter (Node 1), Hot food counter (Node 2), and the till (Node 3): Customers wanting only hot food arrive at a rate of 12 per hour; customers wanting cold food arrive at a rate of 18 per hour; 30% of all customer who buy cold food also want to buy hot food. On average it takes 1 minute to be served cold food, 2 and a half minutes to be served hot food, and 2 minutes to pay. There is 1 server at the cold food counter, 2 servers at the hot food counter, and 2 servers at the till.

A diagram of the system is shown below:



1. How many customers are served during a typical lunch hour?
2. How long are customers expected to wait at each counter?
3. How long are customers expected to be in the system before leaving with their food?
4. You are provided with one extra member of staff. Where should that member of staff be stationed (the cold food counter, the hot food counter, or the till) to have the greatest effect on the customers' overall waiting times?

Confidence Intervals

Confidence intervals around a ***mean***

$$\bar{x} \pm z \frac{s}{\sqrt{n}}$$

Confidence intervals around a ***proportion***

$$\hat{p} \pm z \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$\bar{x}$ is the sample mean
$\hat{p}$ is the sample proportion
s is the sample standard deviation
n is the number of sample
z is the z-score:

Confidence	z
99%	2.56
95%	1.96
90%	1.64

Supplementary Tasks

1. For each of the tasks above, calculate 95% confidence intervals around any relevant KPIs calculated.

2. The following inequality defines a circle inscribed inside the square with corners (0, 0), (0, 1), (1, 1), (1,0):
$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \leq \frac{1}{4}$$
Use Monte Carlo Simulation to estimate the value of π.
Hint: generate Uniformly distributed points in the square, check if they are inside the circle or not. The proportion of points inside the circle will approach the of the circle's area divided by the square's area, that is $\frac{\pi}{4}$.

3. A drunk man is trying to walk home but cannot remember his way home. Each step he will randomly either take a step to the left, a step forward, or a step to the right. After 100 steps, how far away would he expect to be from the starting point? What is the variance in this distance?

4. A bicycle repair shop would like to reconfigure in order to guarantee that all bicycles processed take a maximum of 30 minutes. Their current set up is as follows:
   - Bicycles arrive randomly at the shop at a rate of 15 per hour;
   - They wait in line to be seen at an inspection counter, staffed by one member of staff who can inspect one bicycle at a time. On average an inspection takes around 3 minutes;
   - Around 20% of bicycles do not need repair after inspection, and they are ready for collection;
   - Around 80% of bicycles go on to be repaired after inspection. These then wait in line outside the repair workshop, which is staffed by two members of staff who can each repair one bicycle at a time. On average a repair takes around 6 minutes.

   The workshop will hire one extra member of staff to ensure that most bicycles are completed within 30 minutes. Should that member of staff work on the inspection desk or in the repair workshop?

5. Imagine a manufacturing plant that produces stools:
   - Every 4 seconds a seat arrives on a conveyor-belt.
   - The belt contains three workstations.
   - At each workstation a leg is connected.
   - Connecting a leg takes a random amount of time between 3 seconds and 5 seconds.
   - Between workstations (and before the first workstation) the conveyor-belt is only long enough to hold 3 stools.
   - If the belt before the first workstation is full then new stools fall to the floor and break.
   - If a stool finishes 'service' at a workstation, but there is no space on the conveyor-belt, that stool must remain at the workstation until room becomes available on the conveyor-belt. While this blockage happens, that workstation cannot begin assembling any more stools.

   Each broken stool costs the factory 10p in wasted wood. We wish to know how many stools will fall to the floor and break per hour of operation, and thus the average cost per hour.