Markov Modelling

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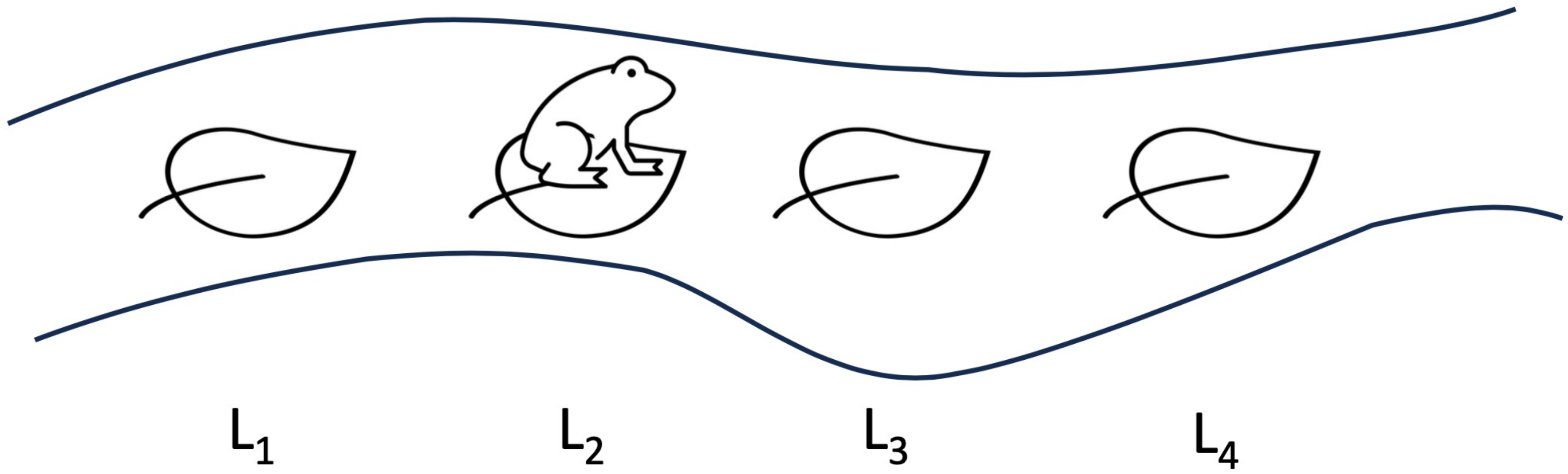
FEB UBB – Cardiff University: International Visiting Lecture Series

1) Stochastic Processes

Motivation



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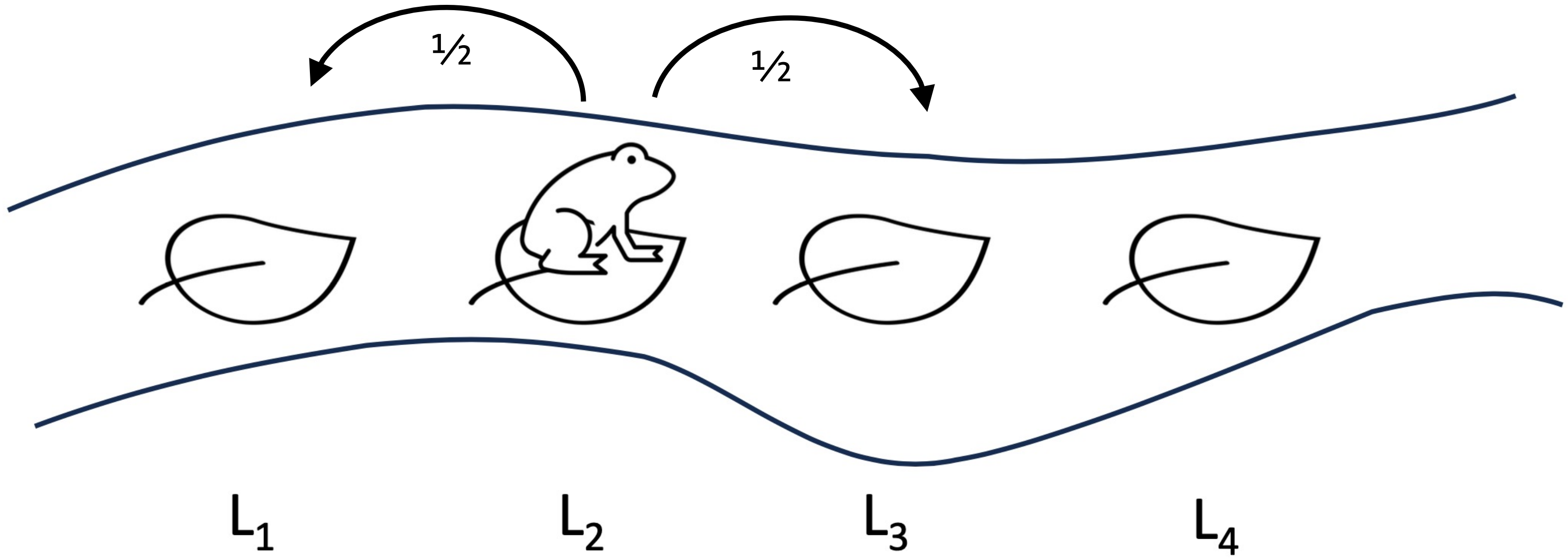

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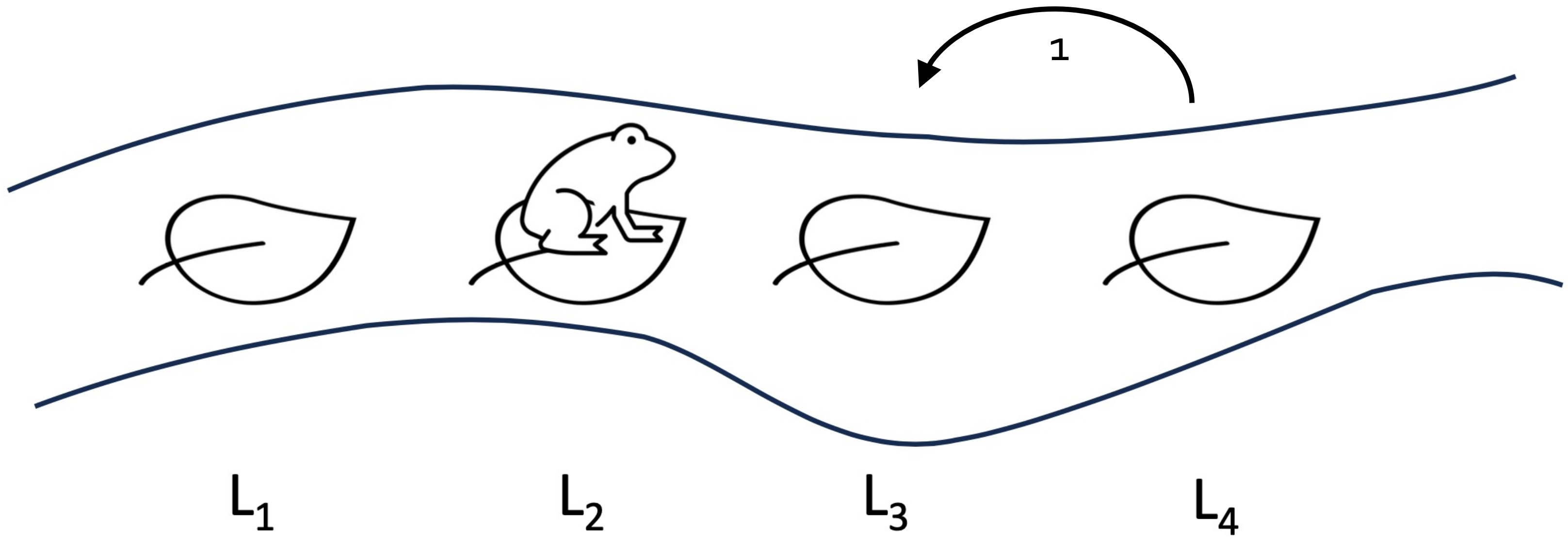

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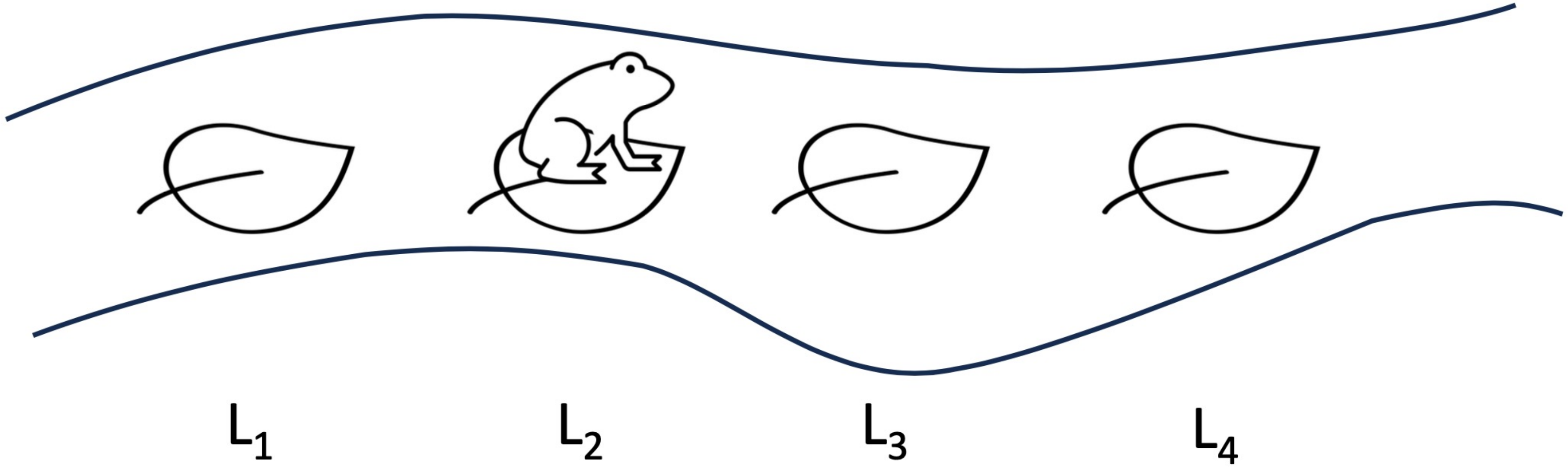

Instance	t=1	t=2	t=3	t=4	t=5	t=6	t=7	t=8	t=9	
Instance 1	L_2	L_3	L_2	L_3	L_4	L_3	L_2	L_3	L_4	…
Instance 2	L_2	L_1	L_2	L_3	L_2	L_1	L_2	L_3	L_4	…
Instance 3	L_2	L_1	L_2	L_1	L_2	L_3	L_4	L_3	L_2	…
…										

Discrete-Time Stochastic Process

A discrete-time stochastic process is a random vector, whose components are indexed by time. It can also be thought of as a family of random variables:

$$\{X_n,\ n \in \mathbb{N}\}$$

Examples?

Markov Property (Discrete-Time)

The Markov property states that X_{n+1} depends only on X_n, and not on any previous states. That is:

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \ldots X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

2) Markov Chains

A discrete time stochastic process satisfying the *Markov property*.

Usually with a countable state space.

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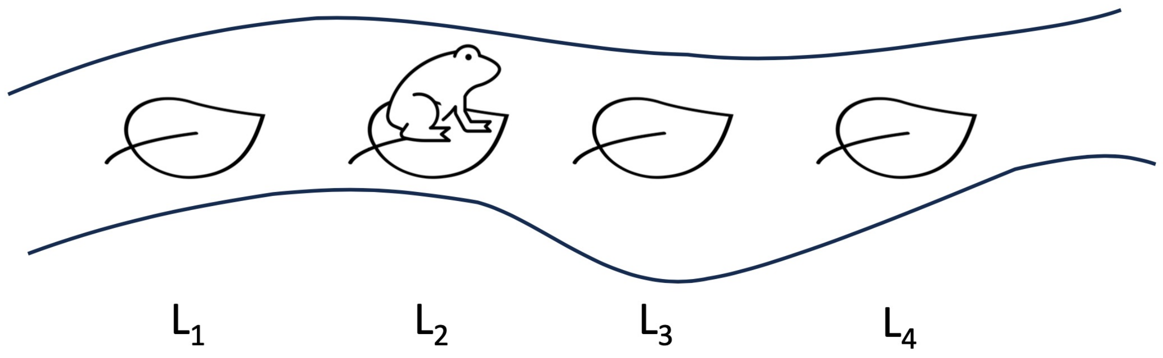


States:

$$S = \{L_1, L_2, L_3, L_4\}$$

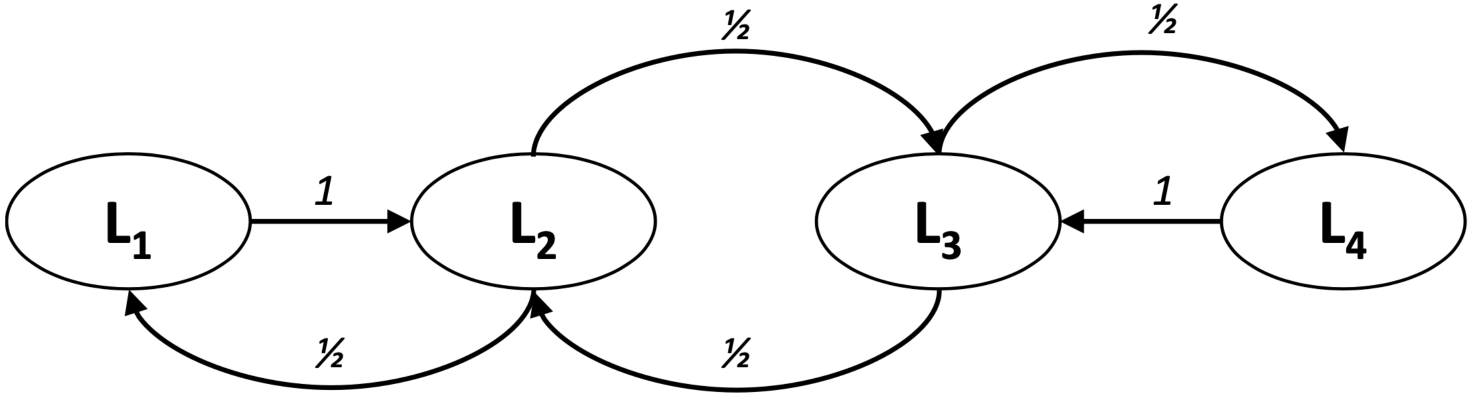
Transition Probability ½ Matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Visualising the Markov Chain

States:

$S = \{L_1, L_2, L_3, L_4\}$



Transition Probability ½ Matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ ½ & 0 & ½ & 0 \\ 0 & ½ & 0 & ½ \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3) Transient Analysis

Chapman-Kolmogorov Equations

Let $\underline{\pi}_n$ be the vector of state probabilities at time n. Then:

DTMC Transitive States

$$\underline{\pi}_{n+1} = \underline{\pi}_n P$$

and by repeated application of this equation:

$$\underline{\pi}_{n+1} = \underline{\pi}_0 P^n$$

Wolfram Alpha

We will solve using Wolfram Alpha:

http://wolframalpha.com/

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Transition Probability ½ Matrix:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ ½ & 0 & ½ & 0 \\ 0 & ½ & 0 & ½ \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The frog is currently on L2. What is the probability that it will be on the L3 in:

1. One time step?
2. Two time steps?
3. Three time steps?

Example *The weather in Cardiff can be either Sunny, Cloudy or Raining:*

- *When Sunny, the probability of it remaining sunny the next day is 0.2, of it clouding over is 0.6, and of it raining is 0.2;*
- *When Cloudy, the probability of it remaining cloudy the next day is 0.5, the probability of the sun coming out is 0.1, and the probability of it raining is 0.4;*
- *When Raining, the probability of it continuing to rain the next day is 0.3, of the sun coming out is 0.3, and of it being cloudy is 0.4.*

Draw the Markov chain and give the transition probability matrix.

Mari is getting married in Cardiff in three days time. The weatherman says that tomorrow there will be 50% chance of sun, 50% chance of cloud, and 0% chance of it raining. What is the probability that it will rain on Mari's wedding day?

TUTORIAL 1

https://www.geraintianpalmer.org.uk/outreach/markov/tutorial_1.pdf

4) Long Run Probabilities

$$P = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

If $\underline{\pi}_0 = (0.5, 0.5, 0)$:

Find:

a) $\underline{\pi}_{65}$

$$P = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

If $\underline{\pi}_0 = (0.5, 0.5, 0)$:

Find:

a) $\underline{\pi}_{65}$

b) $\underline{\pi}_{66}$

$$P = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

If $\underline{\pi}_0 = (0.5, 0.5, 0)$:

Find:

a) $\underline{\pi}_{65}$

b) $\underline{\pi}_{66}$

If $\underline{\pi}_0 = (0.1, 0.3, 0.6)$

Find:

c) $\underline{\pi}_{65}$

$$P^n = P^{n+1} = \lim_{n\to\infty} P^n$$

$$\underline{\pi} = \underline{\pi} P$$

5) Steady-State

DTMC Steady-State

We can find the steady-state probabilities, π of a Markov chain by solving:

$$\underline{\pi} = \underline{\pi} P \tag{2.1}$$

along with the extra constraint enforcing $\underline{\pi}$ to be a valid probability vector:

$$\sum \underline{\pi} = 1$$

$$P = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{pmatrix}$$

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$$\underline{\pi} = \underline{\pi} P$$

$$\sum \underline{\pi} = 1$$

$$\underline{\pi} = (x, y, z)$$

$$x = 0.4x + 0.2y + 0.0z$$
$$y = 0.3x + 0.6y + 0.1z$$
$$z = 0.3x + 0.2y + 0.9z$$
$$x + y + z = 1$$

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$$x + y + z = 1$$

We can also solve using Wolfram Alpha:

http://wolframalpha.com/

Example *Three children, Amir, Benicio, and Carys, are playing pass-the-parcel. They are sat in alphabetical order clockwise. At each beat of the music, the child will pass the parcel to their left clockwise, or keep the parcel for one more beat. Amir will keep the parcel with probability 0.2, Benicio will keep the parcel with probability 0.3, and Carys will keep the parcel with probability 0.4.*
After enough time has passed to assume steady-state, if we randomly switch of the music, what is the probability that each child will be holding the parcel?

TUTORIAL 2

https://www.geraintianpalmer.org.uk/outreach/markov/tutorial_2.pdf

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