Markov Chain Modelling: Solutions 1

Dr Geraint Palmer-Liyu & Dr Syaribah Noor Brice

Tuesday 31st March 2026

1. Consider the Markov chain defined on four states $\{A_1, A_2, A_3, A_4\}$ and transition probability matrix:

$$P = \begin{pmatrix} x & 0.5 & 0.0 & 0.0 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.1 & 0.25 & y & 0.45 \\ 0.2 & z & 0.7 & 0.1 \end{pmatrix}$$

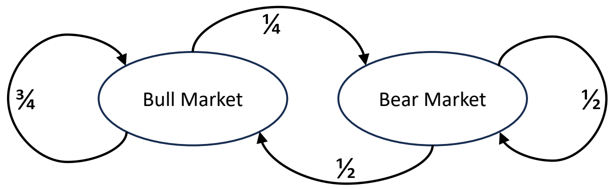
   What values must x, y, and z take for P to be a valid transition probability matrix?

Solution

In order to be a valid transition probability matrix, each row must sum to 1. Therefore:

$$x = 0.5$$
$$y = 0.2$$
$$z = 0.0$$

2. A country's economy can be described as either a Bull market (where stock prices rise and things are going well), or a Bear market (where stock prices fall and things are not going so well). The economy is categorised as such each quarter. This process can be described as a discrete-time Markov chain, with probabilities of being in each state in the next quarter:



If the country is currently in a Bull market, what is the probability of being in either a Bull or a Bear marker in three quarters times?

Solution

We have: $\pi_0 = (1, 0)$, and:

$$P = \begin{pmatrix} {}^{3}\!/_{4} & {}^{1}\!/_{4} \\ {}^{1}\!/_{2} & {}^{1}\!/_{3} \end{pmatrix}$$

We want to find $\pi_3 = \pi_0 P^3$, so:

$$P^3 = \begin{pmatrix} {}^{43}\!/_{64} & {}^{21}\!/_{64} \\ {}^{21}\!/_{32} & {}^{11}\!/_{32} \end{pmatrix}$$

And so:

$$\pi_3 = (1, 0) \begin{pmatrix} {}^{43}\!/_{64} & {}^{21}\!/_{64} \\ {}^{21}\!/_{32} & {}^{11}\!/_{32} \end{pmatrix} = ({}^{43}\!/_{64}, {}^{21}\!/_{64})$$

3. You own a banana farm. Each month, the banana farm can either produce a good yield, a bad yield, or no yield:

   - On a month where a good yield is produced, you can harvest the bananas and sell them for a good price. Once harvested, there are no bananas left.
   - On a month where a bad yield is produced, you do not harvest, and wait until a better yield is produced.
   - On a month with no yield, it is impossible to harvest, and so again you must wait.

   Therefore you always wait for a good yield of bananas.

   It takes one month to grow a bad yield from no yield; and it takes one month to grow a good yield from a bad yield.

   However, you have a monkey problem! Each month there is a probability of 0.1 that the monkeys eat your newly grown bananas. Therefore:

- there is a probability 0.1 that a bad yield remains bad after a month;
- there is a probability 0.1 that no yield remains without any yield after a month;
- and a probability 0.1 that the month after a good yield the farm produces no yield.

Draw the Markov chain for this scenario, and give the transition probability matrix.

This month you observed a bad yield. What is the probability of having a good yeild and therefore making a profit in:

(a) 1 month's time?

(b) 2 month's time?

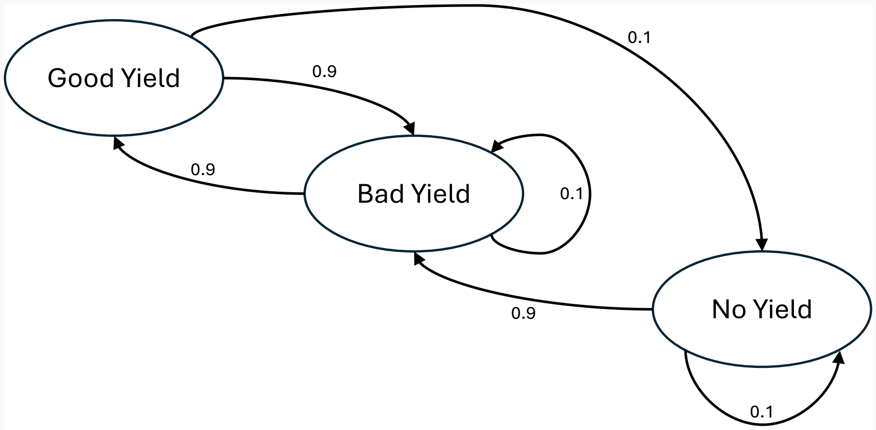
(c) 3 month's time?

Solution

The states that the banana farm can be in at the start of each month are: $\{\text{Good Yield}, \text{Bad Yield}, \text{No Yield}\}$.
When in the bad or no yield state, you do not harvest, and you will advance to either good or bad yield, respectively, *if* no monkeys eat the bananas; so with probability 90%. If they eat the bananas, with probability 10%, the new crops are eaten, and the farm remains in the same state.
When in a state of good yield, you harvest the crops immediately, making profit, and ending up in the state of no yield. In a month's time, your will be in a bad yield state if no monkeys eat the bananas (90% of the time), and remain in a state of no yield if the monkeys do eat them (10% of the time).
Therefore, retaining the order of the states, the transition probability matrix is:

$$P = \begin{pmatrix} 0 & 0.9 & 0.1 \\ 0.9 & 0.1 & 0 \\ 0 & 0.9 & 0.1 \end{pmatrix}$$

Drawing the Markov chain gives:



This month you observe a bad yield, and so $\pi_0 = (0, 1, 0)$. Now:

(a) $\pi_1 = (0, 1, 0)P = (0.9, 0.1, 0)$ and so a 90% chance of making profit in one month's time.

(b) $\pi_2 = (0, 1, 0)P^2 = (0.09, 0.82, 0.09)$ and so a 9% chance of making profit in two months' time.

(c) $\pi_3 = (0, 1, 0)P^3 = (0.738, 0.244, 0.018)$ and so a 73.8% chance of making profit in three months' time.