# Markov Chain Modelling: Solutions 2

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1. Consider the banana farm scenario from earlier.
   (a) What is the steady-state distribution?
   (b) In steady-state, what is the probability of making a profit?

**Solution**

(a) To find steady state we solve $\underline{\pi} = \underline{\pi}P$ and $\sum \underline{\pi} = 1$:

$$\pi_1 = 0.9\pi_2 \tag{1}$$
$$\pi_2 = 0.9\pi_1 + 0.1\pi_2 + 0.9\pi_3 \tag{2}$$
$$\pi_3 = 0.1\pi_1 + 0.1\pi_3 \tag{3}$$
$$1 = \pi_1 + \pi_2 + \pi_3 \tag{4}$$

From Equation 1 we have that $\pi_2 = \frac{10}{9}\pi_1$.
From Equation 3 we have that:

$$\begin{aligned} \pi_3 &= 0.1\pi_1 + 0.1\pi_3 \\ 0.9\pi_3 &= 0.1\pi_1 \\ \pi_3 &= \frac{1}{9}\pi_1 \end{aligned}$$

And therefore we have each variable in terms of multiples of $\pi_1$, so taking Equation 4 we get:

$$\begin{aligned}\pi_1 + \pi_2 + \pi_3 &= 1\\ \pi_1 + \frac{10}{9}\pi_1 + \frac{1}{9}\pi_1 &= 1\\ \frac{20}{9}\pi_1 &= 1\\ \pi_1 &= \frac{9}{20}\end{aligned}$$

And so from the previous relationships we get: $\underline{\pi} = (\pi_1, \pi_2, \pi_3) = (9/20, 1/2, 1/20)$.

(b) Now we only make a profit on months with a good yield. Therefor the probability of making a profit is $\pi_1 = 9/20$

2. A mental health doctor is trying to understand a patient's mental state. They ask the patient to record daily whether they feel Calm, Mildly Anxious, or Very Anxious. Crunching the data the doctor finds:

   - On a calm day, $1/3$ of the time they will remain calm tomorrow, and $2/3$ of the time they will become mildly anxious tomorrow;
   - On a mildly anxious day, $1/4$ of the time they will become calm tomorrow, $1/2$ the time they remain mildly anxious tomorrow, while $1/4$ of the time they become very anxious tomorrow;
   - On a very anxious day, only $1/5$ of the time will they become calm tomorrow, $2/5$ of the time they will become mildly anxious, however $2/5$ of the time they remain very anxious tomorrow.

   (a) Draw the discrete-time Markov chain describing the patient's mental state.

   (b) Find the steady-state probabilities.

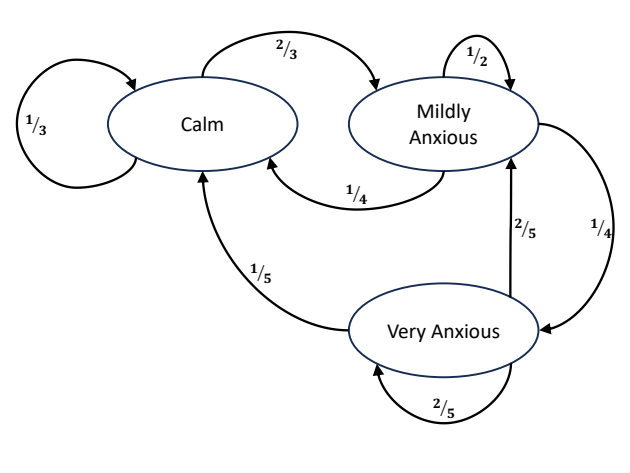
   (c) The doctor devises a medication plan: on calm days the patient should not take any medication; on mildly anxious days they should take a pill of type A, costing $1p$ per pill; and on very anxious days they should take a pill of type B, costing $23$p per pill. What is the expected yearly cost for this medication plan?

Solution

Ordering the states 1-Calm, 2-Mildly Anxious, then 3-Very Anxious, we have:

$$P = \begin{pmatrix} 1/3 & 2/3 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/5 & 2/5 & 2/5 \end{pmatrix}$$

(a) Visualising the Markov chain:




(b) To find steady state we solve $\underline{\pi} = \underline{\pi}P$ and $\sum \underline{\pi} = 1$:

$$\pi_1 = \frac{1}{3}\pi_1 + \frac{1}{4}\pi_2 + \frac{1}{5}\pi_3 \tag{5}$$

$$\pi_2 = \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \tag{6}$$

$$\pi_3 = \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \tag{7}$$

$$1 = \pi_1 + \pi_2 + \pi_3 \tag{8}$$

We can solve for $\pi_3$ in Equation 7:

$$\begin{aligned} \pi_3 &= \frac{1}{4}\pi_2 + \frac{2}{5}\pi_3 \\ \frac{3}{5}\pi_3 &= \frac{1}{4}\pi_2 \\ \pi_3 &= \frac{5}{12}\pi_2 \end{aligned}$$

And solve for $\pi_2$ in Equation 6:

$$\begin{aligned}
\pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\pi_3 \\
\pi_2 &= \frac{2}{3}\pi_1 + \frac{1}{2}\pi_2 + \frac{2}{5}\left(\frac{5}{12}\pi_2\right) \\
\pi_2 &= \frac{2}{3}\pi_1 + \pi_2\left(\frac{1}{2} + \frac{1}{6}\right) \\
\frac{1}{3}\pi_2 &= \frac{2}{3}\pi_1 \\
\pi_2 &= 2\pi_1
\end{aligned}$$

And finally solve for $\pi_1$ in Equation 8:

$$\begin{aligned}
\pi_1 + \pi_2 + \pi_3 &= 1 \\
\pi_1\left(1 + 2 + \left(2 \times \frac{5}{12}\right)\right) &= 1 \\
\frac{23}{6}\pi_1 &= 1 \\
\pi_1 &= \frac{6}{23}
\end{aligned}$$

Implying that $\underline{\pi} = (6/23, 12/23, 5/23)$.

(c) It will cost $1$p per pill each day they are in state 2, and $23$p per pill each day they are in state 3. That is the yearly cost $C$ is:

$$\begin{aligned}
C &= 365\left(1\pi_2 + 23\pi_3\right)p \\
&= 365\left(12/23 + 23\,5/23\right)p \\
&= 365\left(127/23\right)p \\
&= 2015.43p \\
&= £20.15
\end{aligned}$$