Markov Chain Modelling

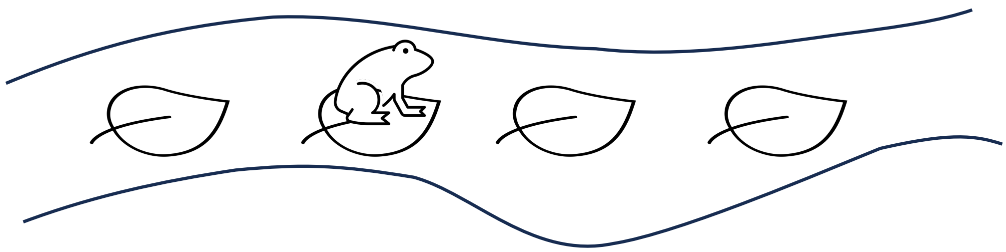
Dr Geraint Palmer-Liyu & Dr Syaribah Noor Brice

Tuesday 31st March 2026

1 Part I: Intro & Transient Analysis

Motivation

It is often possible to describe the behaviour of some stochastic system by describing all the different states that it could be in, and how the system moves between these states. Let's motivate this with an example - consider a river with four lily-pads, and a frog jumping from one to the other randomly every time unit, as shown in the image below. We'll assume that the frog will only jump to an adjacent lily-pad, and will choose between lily-pads with equal probability.



We can describe the system as being in one of four states: either the frog is on the first lily-pad (L_1); the frog is on the second lily-pad (L_2); the frog is on the third lily-pad (L_3); or the frog is on the fourth lily-pad (L_4). This assumes that the time the frog is jumping through the air is instantaneous. The set $S = \{L_1, L_2, L_3, L_4\}$ is called the *state space* of the system.

We also know something about the probability of transitioning from one state to another each time step. Let p_{ij} represent the probability of reaching state j if you are currently in state i, then the following *transition probability matrix* represents all the probabilities of transitioning between all states:

$$P = \begin{array}{c c} & \begin{array}{cccc} L_1 & L_2 & L_3 & L_4 \end{array} \\ \begin{array}{c} L_1 \\ L_2 \\ L_3 \\ L_4 \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

That is p_{L_2,L_3}, denoting that the probability of the frog being on the third lily-pad, given that it is currently on the second lily-pad, is $1/2$.

One realisation of this over 10 time units might be:

Time	1	2	3	4	5	6	7	8	9	10
State	L_1	L_2	L_1	L_2	L_3	L_4	L_3	L_2	L_3	L_2

While another realisation of this over 10 time units might be:

Time	1	2	3	4	5	6	7	8	9	10
State	L_1	L_2	L_3	L_4	L_3	L_2	L_3	L_2	L_1	L_2

We have tools to be able to describe and analysis this random behaviour:

Stochastic Processes

This is an example of a *stochastic process*. In particular, this is an example of a particular type of stochastic process called a discrete-time Markov chain.

Discrete-Time Stochastic Process

A discrete-time stochastic process is a random vector, whose components are indexed by time. It can also be thought of as a family of random variables:

$$\{X_n,\ n \in \mathbb{N}\}$$

A discrete-time Markov chain is a discrete-time stochastic process, where the outcome of each X_n is an element of a countable set of states, called the *state space*, and that satisfies the *Markov property*:

Markov Property (Discrete-Time)

The Markov property states that X_{n+1} depends only on X_n, and not on any previous states. That is:

$$\mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n, X_{n-1} = x_{n-1}, \dots X_0 = x_0) = \mathbb{P}(X_{n+1} = x_{n+1} \mid X_n = x_n)$$

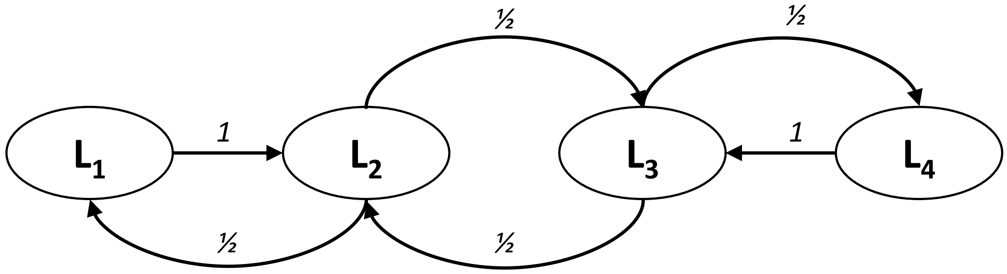
Markov Chains

In this way, we can define our frog leaping example as a Markov chain. We need only define the *state space* and *transition probability matrix*. Note that the state space may be infinite.

- The state space S is a set of states. It is useful to order the states $S = \{s_0, s_1, \dots\}$.
- A valid transition probability matrix, P, has entries $p_{ij} = \mathbb{P}(X_n = s_j \, | X_{n-1} = s_i \,)$, where $0 \leq p_{ij} \leq 1$ which are valid probabilities, and rows that sum to 1.

Visualising Markov Chains

It is often useful to visualise Markov chains. A common way to do this is to represent states in ovals, and transitions as arrows between states, labelled by the probabilities. For example, for the frog-leaping example:



Transient Analysis

The Chapman-Kolmogorov equations, equivalent to matrix multiplication, allow us to find transient probabilities, that is the probability of being in each state in a finite number of time steps, given a particular starting condition. Let $\underline{\pi}_n$ be a vector representing the probabilities of being in each state at time n, and the transition probability matrix is denoted by P, then:

DTMC Transitive States

$$\underline{\pi}_{n+1} = \underline{\pi}_n P$$

and by repeated application of this equation:

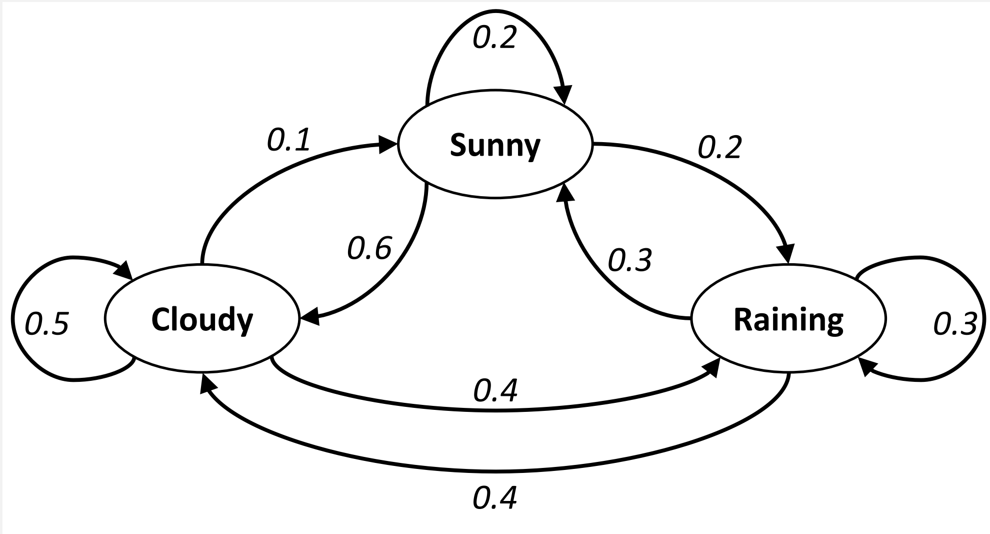
$$\underline{\pi}_{n+1} = \underline{\pi}_0 P^{n+1}$$

Example 1 *The weather in Cardiff can be either Sunny, Cloudy or Raining:*

- *When Sunny, the probability of it remaining sunny the next day is 0.2, of it clouding over is 0.6, and of it raining is 0.2;*
- *When Cloudy, the probability of it remaining cloudy the next day is 0.5, the probability of the sun coming out is 0.1, and the probability of it raining is 0.4;*
- *When Raining, the probability of it continuing to rain the next day is 0.3, of the sun coming out is 0.3, and of it being cloudy is 0.4.*

Draw the Markov chain and give the transition probability matrix.
Mari is getting married in Cardiff in three days time. The weatherman says that tomorrow there will be 50% chance of sun, 50% chance of cloud, and 0% chance of it raining. What is the probability that it will rain on Mari's wedding day?

Solution to Example 1 *Drawing the Markov chain gives:*



The transition probability matrix, ordering the states Sunny, Cloudy, Raining, is:

$$P = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

The weatherman has stated that $\underline{\pi}_1 = (0.5, 0.5, 0.0)$. *Mari is getting married at* $n = 3$*, therefore:*

$$\begin{aligned}
\underline{\pi}_3 &= \underline{\pi}_1 P^2 \\
&= (0.5, 0.5, 0.0) \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}^2 \\
&= (0.5, 0.5, 0.0) \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix} \\
&= (0.5, 0.5, 0.0) \begin{pmatrix} 0.16 & 0.5 & 0.34 \\ 0.19 & 0.47 & 0.34 \\ 0.19 & 0.5 & 0.31 \end{pmatrix} \\
&= (0.175, 0.485, 0.34)
\end{aligned}$$

Therefore there is a 34% chance of raining on Mari's wedding day.

2 Part II: Steady-state Analysis

Long-run Probabilities

We have seen that the rows of P^n gives the probabilities of being in each state in n time steps, given the probability of currently being in each state. It can be seen experimentally, if we take n to be very large, then we end up with a matrix where each row is identical.

For example, consider the weather Markov chain from Example 1:

$$P = \begin{pmatrix} 0.2 & 0.6 & 0.2 \\ 0.1 & 0.5 & 0.4 \\ 0.3 & 0.4 & 0.3 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0.16 & 0.5 & 0.34 \\ 0.19 & 0.47 & 0.34 \\ 0.19 & 0.5 & 0.31 \end{pmatrix}$$

$$\vdots$$

$$P^{100} = \begin{pmatrix} 0.18446602 & 0.48543689 & 0.33009709 \\ 0.18446602 & 0.48543689 & 0.33009709 \\ 0.18446602 & 0.48543689 & 0.33009709 \end{pmatrix}$$

$$P^{101} = \begin{pmatrix} 0.18446602 & 0.48543689 & 0.33009709 \\ 0.18446602 & 0.48543689 & 0.33009709 \\ 0.18446602 & 0.48543689 & 0.33009709 \end{pmatrix} \qquad \vdots$$

We notice two things from this:

- if each row is identical, then is matters not which state we begin in;
- after a some number time steps, $P^n = P^{n+1} = \lim_{n\to\infty} P^n$.

Steady-State Analysis

When this situation occurs, the system is said to be in *steady-state*. The rows of resultant matrix $\lim_{n\to\infty} P^n$ is called the long-run probabilities, or steady-state probabilities of the Markov chain.

Existence of Steady-States
A discrete-time Markov chain consisting of one irreducible class, and all states are aperiodic, is guaranteed to have a unique steady-state distribution.

We can find the steady-state probabilities:

DTMC Steady-State

We can find the steady-state probabilities, π of a Markov chain by solving:

$$\underline{\pi} = \underline{\pi} P \tag{1}$$

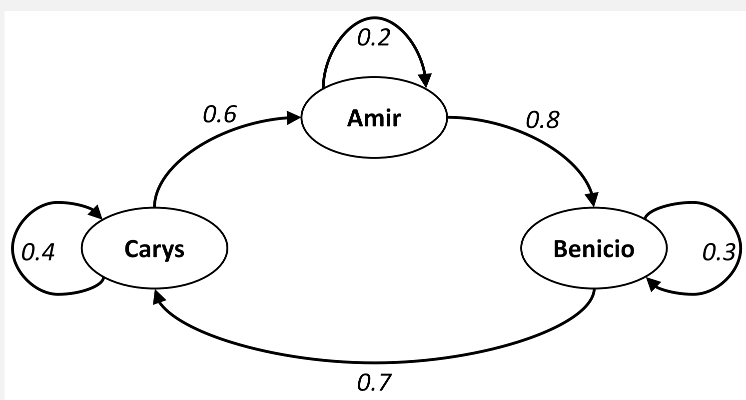
along with the extra constraint enforcing $\underline{\pi}$ to be a valid probability vector:

$$\sum \underline{\pi} = 1$$

We can solve this using various linear algebraic methods.

Example 2 *Three children, Amir, Benicio, and Carys, are playing pass-the-parcel. They are sat in alphabetical order clockwise. At each beat of the music, the child will pass the parcel to their left clockwise, or keep the parcel for one more beat. Amir will keep the parcel with probability 0.2, Benicio will keep the parcel with probability 0.3, and Carys will keep the parcel with probability 0.4.*
After enough time has passed to assume steady-state, if we randomly switch of the music, what is the probability that each child will be holding the parcel?

Solution to Example 2 *Drawing the Markov chain will help:*



We have:

$$P = \begin{pmatrix} 0.2 & 0.8 & 0 \\ 0 & 0.3 & 0.7 \\ 0.6 & 0 & 0.4 \end{pmatrix}$$

Solution to Example 2 (continuing from p. 7) *and we wish to find $\underline{\pi}$ by solving:*

$$\begin{aligned}\underline{\pi} &= \underline{\pi}P \\ \sum \underline{\pi} &= 1\end{aligned}$$

Yielding

$$\begin{aligned}\pi_A &= 0.2\pi_A + 0.6\pi_C \\ \pi_B &= 0.8\pi_A + 0.3\pi_B \\ \pi_C &= 0.7\pi_B + 0.4\pi_C \\ 1 &= \pi_A + \pi_B + \pi_C\end{aligned}$$

The first two equations gives us all components in terms of π_C:

$$\begin{aligned}\pi_A &= \frac{0.6}{1-0.2}\pi_C = \frac{3}{4}\pi_C \\ \pi_B &= \frac{0.8}{1-0.3}\pi_A = \frac{8}{7}\pi_A \\ &= \frac{8}{7}\left(\frac{3}{4}\pi_C\right) \\ &= \frac{6}{7}\pi_C\end{aligned}$$

Substituting into the final equation gives:

$$\begin{aligned}\pi_A + \pi_B + \pi_C &= 1 \\ \frac{3}{4}\pi_C + \frac{6}{7}\pi_C + \pi_C &= 1 \\ \pi_C\left(\frac{3}{4} + \frac{6}{7} + 1\right) &= 1 \\ \frac{73}{28}\pi_C &= 1 \\ \pi_C &= \frac{28}{73}\end{aligned}$$

Solution to Example 2 (continuing from p. 8) *Now as we have all components in terms of* π_C*:*

$$\begin{aligned}
\pi_C &= \frac{28}{73} \\
\pi_B &= \frac{6}{7}\pi_C = \frac{6}{7}\left(\frac{28}{73}\right) = \frac{24}{73} \\
\pi_A &= \frac{3}{4}\pi_C = \frac{3}{4}\left(\frac{28}{73}\right) = \frac{21}{73}
\end{aligned}$$

And so the probabilities that each child will be left holding the parcel is $\underline{\pi} = ({}^{21}/{}_{73}, {}^{24}/{}_{73}, {}^{28}/{}_{72})$.

3 Summary

Here is a summary table comparing notation and concepts of discrete-time Markov chains:

	DTMC
defining transitions	P (matrix of probabilities)
transient probability vector	$\underline{\pi}_n$ (probability vector at time step n)
transient probabilities	$\pi_{i,n}$ (i^{th} element of $\underline{\pi}_n$)
steady-state probability vector	$\underline{\pi}$
i^{th} element of $\underline{\pi}$	π_i
transient probability relationship	$\underline{\pi}_{n+1} = \underline{\pi}_n P$
steady-state relationship	$\underline{\pi} = \underline{\pi} P$