A Brief Introduction to Markov Chains

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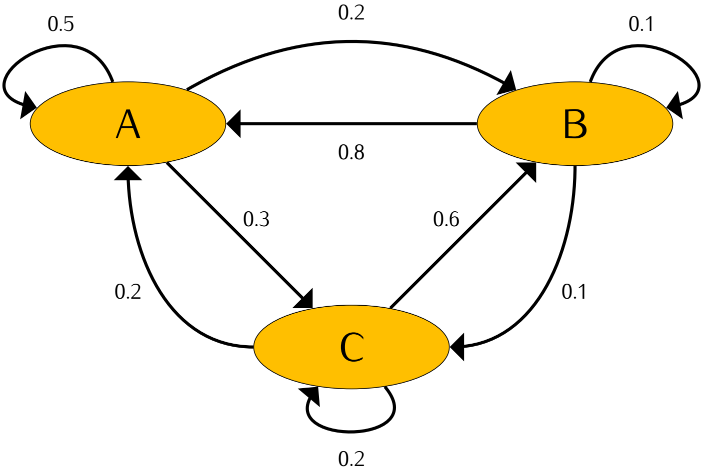
1. Discrete Time Markov Chains
    - Steady-State Probabilities
    - Higher Order Markov Chains
2. Absorbing Markov Chains
    - Snakes & Ladders
3. Continuous Time Markov Chains
    - A Simple Queue

Figure: from Flickr brickset

Andrei Andreyevich Markov

Figure: Markov chain pioneer.

What is a Markov Chain?



$$P = \begin{pmatrix} 0.5 & 0.2 & 0.3 \\ 0.8 & 0.1 & 0.1 \\ 0.2 & 0.6 & 0.2 \end{pmatrix}$$

Initial state $\pi_0 = (1, 0, 0)$:

$$\downarrow$$

$$\pi_{t+1} = \pi_t P$$

$$\pi_t = \pi_0 P^t$$

$$\downarrow$$

$$\pi_3 = (0.521, 0.262, 0.217)$$

Initial state $\pi_0 = (103, 147, 82)$:

$$\downarrow$$

$$\pi_{t+1} = \pi_t P$$

$$\pi_t = \pi_0 P^t$$

$$\downarrow$$

$$\pi_3 = (167.875, 88.725, 75.400)$$

Initial state $\pi_0 = (0.1, 0.3, 0.6)$:

$$\pi_{t+1} = \pi_t P$$

$$\pi_t = \pi_0 P^t$$

$$\pi_3 = (0.5093, 0.2569, 0.2338)$$

Steady-State Probabilities

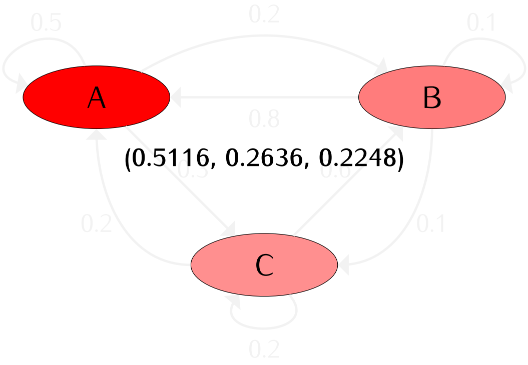
$$\pi = \pi P$$

$$\Sigma \pi = 1$$

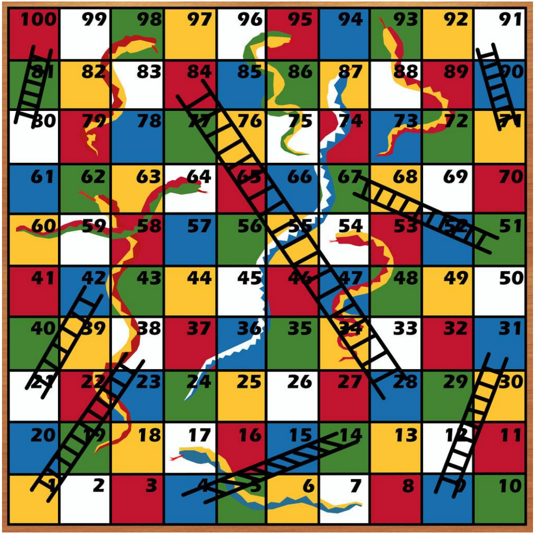
Steady-State Probabilities

$$\pi = \pi P$$

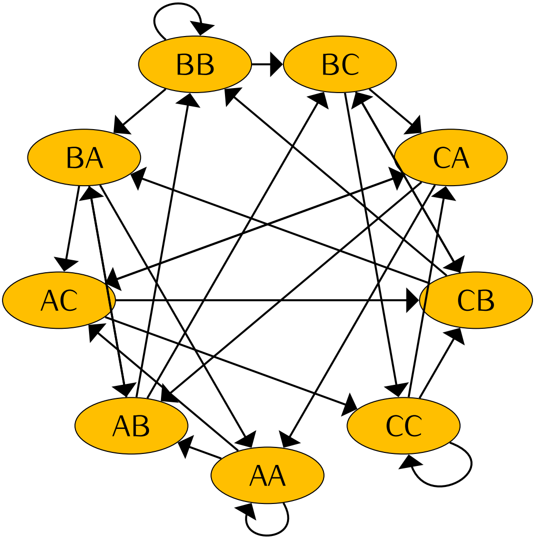
$$\Sigma\pi = 1$$



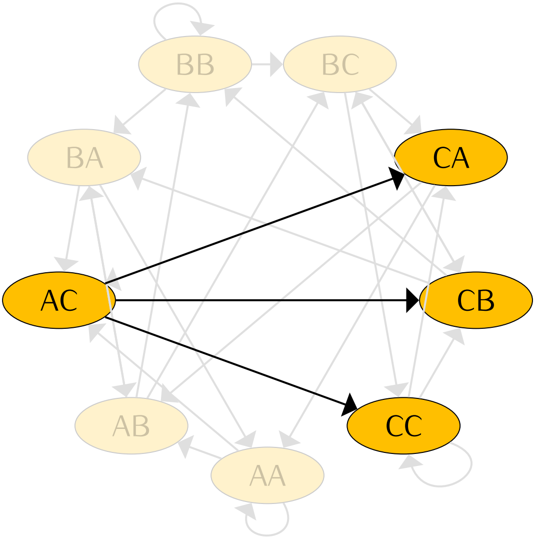
The Markov Property



Higher Order Markov Chains



Higher Order Markov Chains



the

GRAIN
MOTHER
WALL
ZONE
SUCCESS
SHADOW
JOB
T.A.R.D.I.S.
CLOCK
TIDE
DISAPPOINTMENT

against the

GRAIN
MOTHER
WALL
ZONE
SUCCESS
SHADOW
JOB
T.A.R.D.I.S.
CLOCK
TIDE
DISAPPOINTMENT

race against the

GRAIN

MOTHER

WALL

ZONE

SUCCESS

SHADOW

JOB

T.A.R.D.I.S.

CLOCK

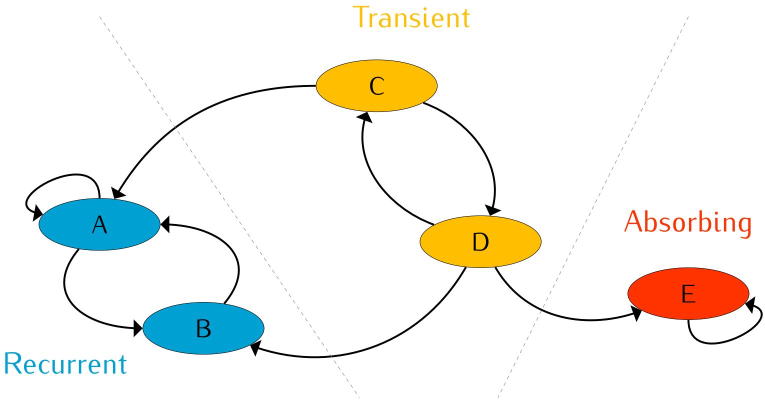
TIDE

DISAPPOINTMENT

Generating Music with Markov Chains

https://www.youtube.com/watch?v=qOZ2Q-Ls48U

Classification of States



Absorbing Markov Chains

Probability of Absorption

$$\mathbb{P}\,(\text{absorption in } t \text{ steps from } s) = P^t_{(s,a)}$$

$$\lim_{t\to\infty} P^t_{(s,a)} \to 1$$

Absorbing Markov Chains

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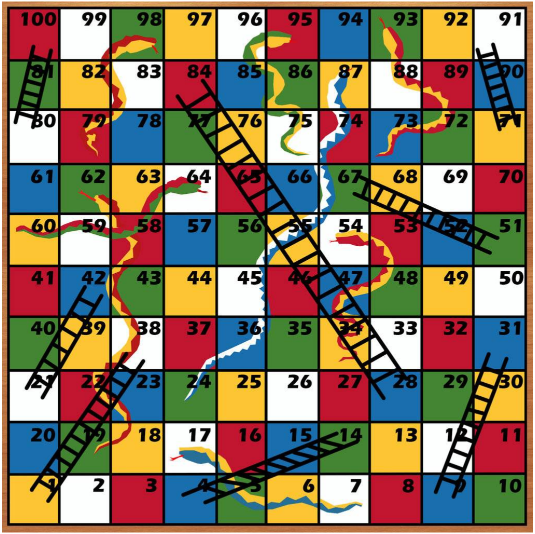
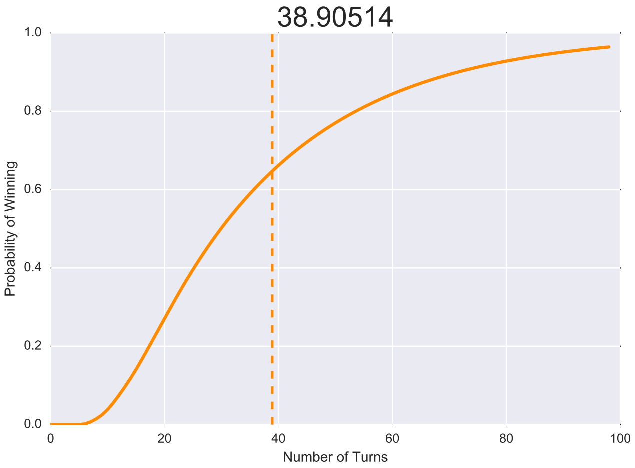
Mean Steps to Absorption

$$P = \begin{pmatrix} Q & R \\ \mathbf{0} & 1 \end{pmatrix}$$

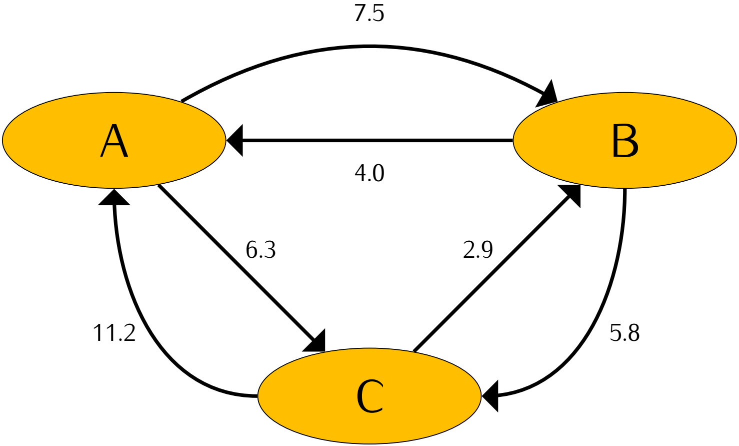
$$\mathbb{E}\left[\text{steps to absorption from } s\right] = \left(\mathbb{I} - Q\right)^{-1}{}_{(s)}$$



100 99 98 97 96 95 94 93 92 91
82 83 84 85 86 87 88 89 90
80 79 78 76 75 74 73 72
61 62 63 64 65 66 67 68 69 70
60 59 58 57 56 55 54 53 51
41 42 43 44 45 47 48 49 50
40 39 38 37 36 35 33 32 31
21 22 23 24 25 26 27 28 29 30
20 19 18 17 16 15 14 13 11
1 2 3 6 7 8 10


38.90514
Probability of Winning
Number of Turns
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100

Continuous-Time Markov Chains

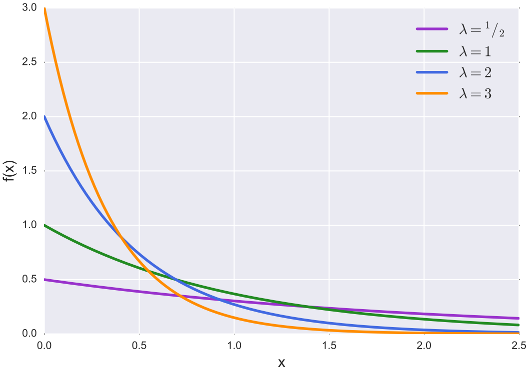


$$Q = \begin{pmatrix} -13.8 & 7.5 & 6.3 \\ 4.0 & -9.8 & 5.8 \\ 11.2 & 2.9 & -14.1 \end{pmatrix}$$

Discrete	Continuous
$\pi_t = \pi_0 P^t$	$\pi_t = \pi_0 \left(\mathbb{I} + \sum_{k=1}^{\infty} \frac{Q^k t^k}{k!} \right)$
$\pi = \pi P$	$0 = \pi Q$

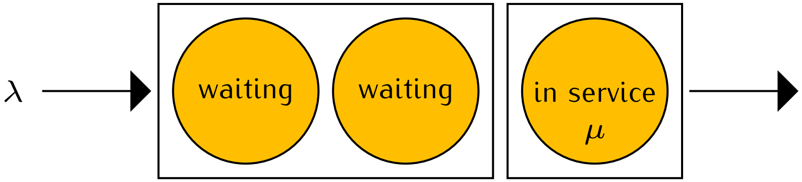
The Exponential Distribution

$$f(x) = \lambda e^{-\lambda x}$$

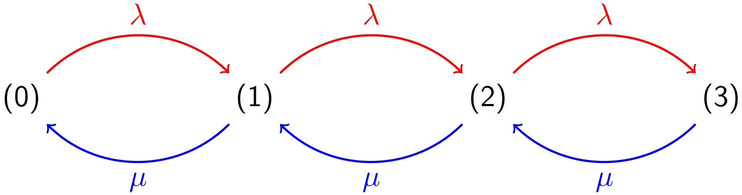


$$\mathbb{P}(T > s + t \mid T > t) = \mathbb{P}(T > s)$$

Modelling a Queue



Arrivals $\sim$ Poisson(λ)
Service time $\sim$ Exponential(μ)



λ
λ
λ
(0)
(1)
(2)
(3)
μ
μ
μ

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{pmatrix}$$

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 \\ \mu & -(\lambda+\mu) & \lambda & 0 \\ 0 & \mu & -(\lambda+\mu) & \lambda \\ 0 & 0 & \mu & -\mu \end{pmatrix}$$

$$\pi_0 = \frac{\mu^3}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

$$\pi_1 = \frac{\lambda\mu^2}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

$$\pi_2 = \frac{\lambda^2\mu}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

$$\pi_3 = \frac{\lambda^3}{\lambda^3 + \lambda^2\mu + \lambda\mu^2 + \mu^3}$$

Thank You!